第 3 章　电阻网络分析

本章列举了一些分析电阻网络的技巧和方法。这一章以电阻网络变量及电阻网络分析的问题作为开始，接着介绍了现在使用最广泛的节点法和网孔分析法。此两种方法是推到全电路方程的两个最普遍的方法。本章试图通过它们在具体电阻网络里的应用来使读者熟悉这两种技巧。另外本章中介绍的基于叠加原理的第二种方法仅适用于线性网络。接着，戴维南和诺顿等效电路的概念被引出来，进而对电路中最大功率传输和非线性负载以及负载线分析展开了讨论。通过本章学习，读者应该有信心可以掌握几种解决电阻网络问题的计算方法。下面的表格列出了学习本章的目的。

学习目标

1. 利用节点法，计算含有线性电阻、独立电源以及非独立电源的电路问题。第 3.2 和 3.4 节。
2. 利用网孔分析法，计算含有线性电阻、独立电源以及非独立电源的电路问题。第 3.3 和 3.4 节。
3. 在含有独立电源的线性电路中，学会使用叠加原理解决问题。
4. 掌握戴维南和诺顿的电路等效代换方法解决含有线性电阻和独立电源的电路。第 3.5 节。
5. 在一个电源和一个电阻之间，采用等效电路替换的思想，计算最大功率输出。第 3.7 节。
6. 通过负载线分析和解析方法，利用等效电路的思想来计算非线性电阻电路的电压、电流和功率。

3.1　电路分析

分析一个电路包括确定每一条支路的电流以及各个节点的电压。因此尽量清楚详细地并系统地定义所有的相关变量就显得尤为重要了。一旦明确了已知和未知的变量，就可以建立一系列的关于这些变量的方程，并通过适当的方法解这些方程。电路分析包括建立一个最小的方程组但能求出所有的未知变量。写出这些方程的步骤是第 3 章主要探讨的话题。本书将这些步骤归纳成了简单法则的形式。按照标准的规则进行分析，分析电路的过程被大大的简化。

例题 3.1 定义了与一个特定电路有关的所有的电压和电流。

例题 3.1

问题：如图 3.1，分析电路中各支路和节点电压以及各回路和网孔的电流。

解答：以下的节点电压可以定义为：

节 点 电 压	支 路 电 压
$v_a = v_s$（电源电压）	$v_s = v_a - v_d = v_a$
$v_b = v_{R2}$	$v_{R1} = v_a - v_b$
$v_c = v_{R4}$	$v_{R2} = v_b - v_d = v_b$
$v_d = 0$（地）	$v_{R3} = v_b - v_c$
	$v_{R4} = v_c - v_d = v_c$

注释：电流 i_a、i_b、i_c 是回路电流，但是只有 i_a、i_b 是网孔电流。

CHAPTER

3

RESISTIVE NETWORK ANALYSIS

Chapter 3 illustrates the fundamental techniques for the analysis of resistive circuits. The chapter begins with the definition of network variables and of network analysis problems. Next, the two most widely applied methods—*node analysis* and *mesh analysis*—are introduced. These are the most generally applicable circuit solution techniques used to derive the equations of all electric circuits; their application to resistive circuits in this chapter is intended to acquaint you with these methods, which are used throughout the book. The second solution method presented is based on the *principle of superposition,* which is applicable only to linear circuits. Next, the concept of *Thévenin and Norton equivalent circuits* is explored, which leads to a discussion of *maximum power transfer* in electric circuits and facilitates the ensuing discussion of nonlinear loads and *load-line analysis.* At the conclusion of the chapter, you should have developed confidence in your ability to compute numerical solutions for a wide range of resistive circuits. The following box outlines the principal learning objectives of the chapter.

Learning Objectives

1. Compute the solution of circuits containing linear resistors and independent and dependent sources by using *node analysis. Sections 3.2 and 3.4.*
2. Compute the solution of circuits containing linear resistors and independent and dependent sources by using *mesh analysis. Sections 3.3 and 3.4.*
3. Apply the *principle of superposition* to linear circuits containing independent sources. *Section 3.5.*
4. Compute *Thévenin and Norton equivalent circuits* for networks containing linear resistors and independent and dependent sources. *Section 3.6.*
5. Use equivalent-circuit ideas to compute the *maximum power transfer* between a source and a load. *Section 3.7.*
6. Use the concept of equivalent circuit to determine voltage, current, and power for nonlinear loads by using *load-line analysis* and analytical methods. *Section 3.8.*

3.1 Network Analysis

The analysis of an electric network consists of determining each of the unknown branch currents and node voltages. It is therefore important to define all the relevant variables as clearly as possible, and in systematic fashion. Once the known and unknown variables have been identified, a set of equations relating these variables is constructed, and these equations are solved by means of suitable techniques. The analysis of electric circuits consists of writing the smallest set of equations sufficient to solve for all the unknown variables. The procedures required to write these equations are the subject of Chapter 3 and are very well documented and codified in the form of simple rules. The analysis of electric circuits is greatly simplified if some standard conventions are followed.

Example 3.1 defines all the voltages and currents that are associated with a specific circuit.

EXAMPLE 3.1

Problem

Identify the branch and node voltages and the loop and mesh currents in the circuit of Figure 3.1.

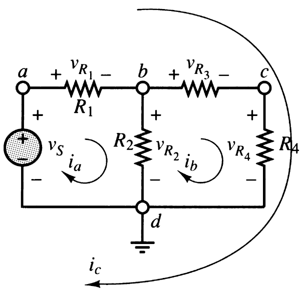


Figure 3.1

Solution

The following node voltages may be identified:

Node voltages	Branch voltages
$v_a = v_S$ (source voltage)	$v_S = v_a - v_d = v_a$
$v_b = v_{R_2}$	$v_{R_1} = v_a - v_b$
$v_c = v_{R_4}$	$v_{R_2} = v_b - v_d = v_b$
$v_d = 0$ (ground)	$v_{R_3} = v_b - v_c$
	$v_{R_4} = v_c - v_d = v_c$

Comments: Currents i_a, i_b, and i_c are loop currents, but only i_a and i_b are mesh currents.

In the example, we have identified a total of 9 variables! It should be clear that some method is needed to organize the wealth of information that can be generated simply by applying Ohm's law at each branch in a circuit. What would be desirable at this point is a means of reducing the number of equations needed to solve a circuit to the minimum necessary, that is, a method for obtaining N equations in N unknowns. The remainder of the chapter is devoted to the development of systematic circuit analysis methods that will greatly simplify the solution of electrical network problems.

3.2 THE NODE VOLTAGE METHOD

Node voltage analysis is the most general method for the analysis of electric circuits. In this section, its application to linear resistive circuits is illustrated. The **node voltage method** is based on defining the voltage at each node as an independent variable. One of the nodes is selected as a **reference node** (usually—but not necessarily—ground), and each of the other node voltages is referenced to this node. Once each node voltage is defined, Ohm's law may be applied between any two adjacent nodes to determine the current flowing in each branch. In the node voltage method, *each branch current is expressed in terms of one or more node voltages;* thus, currents do not explicitly enter into the equations. Figure 3.2 illustrates how to define branch currents in this method. You may recall a similar description given in Chapter 2.

Once each branch current is defined in terms of the node voltages, Kirchhoff's current law is applied at each node:

$$\sum i = 0 \tag{3.1}$$

Figure 3.3 illustrates this procedure.

In the node voltage method, we assign the node voltages v_a and v_b; the branch current flowing from a to b is then expressed in terms of these node voltages.

$$i = \frac{v_a - v_b}{R}$$

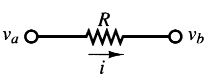


Figure 3.2 Branch current formulation in node analysis

By KCL: $i_1 - i_2 - i_3 = 0$. In the node voltage method, we express KCL by

$$\frac{v_a - v_b}{R_1} - \frac{v_b - v_c}{R_2} - \frac{v_b - v_d}{R_3} = 0$$

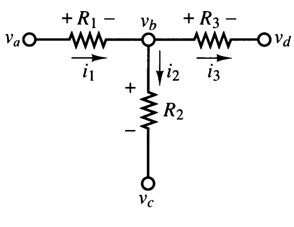
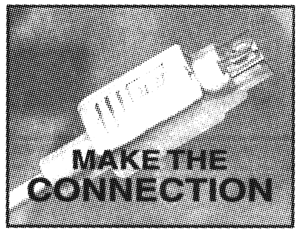


Figure 3.3 Use of KCL in node analysis

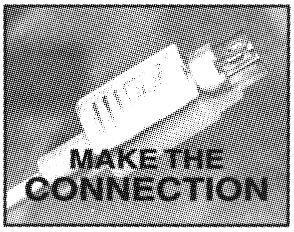
The systematic application of this method to a circuit with n nodes leads to writing n linear equations. However, one of the node voltages is the reference voltage and is therefore already known, since it is usually assumed to be zero (recall that the choice of reference voltage is dictated mostly by convenience, as explained in Chapter 2). Thus, we can write $n - 1$ *independent linear equations* in the $n - 1$ independent variables (the node voltages). Node analysis provides the minimum number of equations required to solve the circuit, since any branch voltage or current may be determined from knowledge of node voltages.



Thermal Systems

A useful analogy can be found between electric circuits and thermal systems. The table below illustrates the correspondence between electric circuit variables and thermal system variables, showing that the difference in electrical potential is analogous to the temperature difference between two bodies. Whenever there is a temperature difference between two bodies, Newton's law of cooling requires that heat flow from the warmer body to the cooler one. The flow of heat is therefore analogous to the flow of current. Heat flow can take place based on one of three mechanisms: (1) conduction, (2) convection, and (3) radiation. In this sidebar we only consider the first two, for simplicity.

Electrical variable	Thermal variable
Voltage difference v, [V]	Temperature difference ΔT, [°C]
Current i, A	Heat flux q, [W]
Resistance R, [Ω/m]	Thermal resistance R_t [°C/W]
Resistivity ρ, [Ω/m]	Conduction heat-transfer coefficient k, $\left[\frac{W}{m - °C}\right]$
(No exact electrical analogy)	Convection heat-transfer coefficient, or film coefficient of heat-transfer h, $\left[\frac{W}{m^2 - °C}\right]$

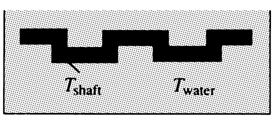


Thermal Resistance

To explain thermal resistance, consider a heat treated engine crankshaft that has just completed some thermal treatment. Assume that the shaft is to be quenched in a water bath at ambient temperature (see the figure below). Heat flows from within the shaft to the surface of the shaft, and then from the shaft surface to the water. This process continues until the temperature of the shaft is equal to that of the water.

The first mode of heat transfer in the above description is called *conduction*, and it occurs because the thermal conductivity of steel causes heat to flow from the higher temperature inner core to the lower-temperature surface. The heat transfer conduction coefficient k is analogous to the resistivity ρ of an electric conductor.

The second mode of heat transfer, *convection*, takes place at the boundary of two dissimilar materials (steel and water here). Heat transfer between the shaft and water is dependent on the surface area of the shaft in contact with the water A and is determined by the heat transfer convection coefficient h.



Engine crankshaft quenched in water bath.

The node analysis method may also be defined as a sequence of steps, as outlined in the following box:

FOCUS ON METHODOLOGY

NODE VOLTAGE ANALYSIS METHOD

1. Select a reference node (usually ground). This node usually has most elements tied to it. All other nodes are referenced to this node.
2. Define the remaining $n - 1$ node voltages as the independent or dependent variables. Each of the m voltage sources in the circuit is associated with a dependent variable. If a node is not connected to a voltage source, then its voltage is treated as an independent variable.
3. Apply KCL at each node labeled as an independent variable, expressing each current in terms of the adjacent node voltages.
4. Solve the linear system of $n - 1 - m$ unknowns.

Following the procedure outlined in the box guarantees that the correct solution to a given circuit will be found, provided that the nodes are properly identified and KCL is applied consistently. As an illustration of the method, consider the circuit shown in Figure 3.4. The circuit is shown in two different forms to illustrate equivalent graphical representations of the same circuit. The circuit on the right leaves no question where the nodes are. The direction of current flow is selected arbitrarily (assuming that i_S is a positive current). Application of KCL at node a yields

$$i_S - i_1 - i_2 = 0 \tag{3.2}$$

whereas at node b

$$i_2 - i_3 = 0 \tag{3.3}$$

It is instructive to verify (at least the first time the method is applied) that it is not necessary to apply KCL at the reference node. The equation obtained at node c,

$$i_1 + i_3 - i_S = 0 \tag{3.4}$$

is not independent of equations 3.2 and 3.3; in fact, it may be obtained by adding the

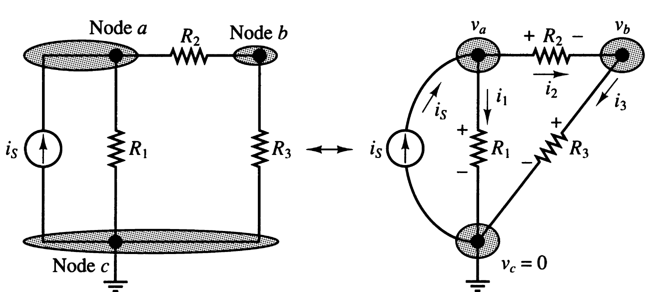


Figure 3.4 Illustration of node analysis

equations obtained at nodes a and b (verify this, as an exercise). This observation confirms the statement made earlier:

In a circuit containing n nodes, we can write at most $n - 1$ independent equations.

Now, in applying the node voltage method, the currents i_1, i_2, and i_3 are expressed as functions of v_a, v_b, and v_c, the independent variables. Ohm's law requires that i_1, for example, be given by

$$i_1 = \frac{v_a - v_c}{R_1} \tag{3.5}$$

since it is the potential difference $v_a - v_c$ across R_1 that causes current i_1 to flow from node a to node c. Similarly,

$$\begin{aligned} i_2 &= \frac{v_a - v_b}{R_2} \\ i_3 &= \frac{v_b - v_c}{R_3} \end{aligned} \tag{3.6}$$

Substituting the expression for the three currents in the nodal equations (equations 3.2 and 3.3), we obtain the following relationships:

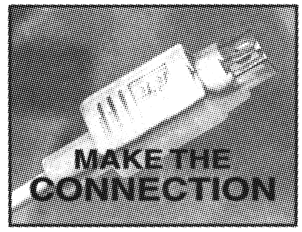
$$i_S - \frac{v_a}{R_1} - \frac{v_a - v_b}{R_2} = 0 \tag{3.7}$$

$$\frac{v_a - v_b}{R_2} - \frac{v_b}{R_3} = 0 \tag{3.8}$$

Equations 3.7 and 3.8 may be obtained directly from the circuit, with a little practice. Note that these equations may be solved for v_a and v_b, assuming that i_S, R_1, R_2, and R_3 are known. The same equations may be reformulated as follows:

$$\begin{aligned} \left(\frac{1}{R_1} + \frac{1}{R_2}\right) v_a + \left(-\frac{1}{R_2}\right) v_b &= i_S \\ \left(-\frac{1}{R_2}\right) v_a + \left(\frac{1}{R_2} + \frac{1}{R_3}\right) v_b &= 0 \end{aligned} \tag{3.9}$$

Examples 3.2 through 3.4 further illustrate the application of the method.



Thermal Circuit Model

The *conduction resistance* of the shaft is described by the following equation:

$$q = \frac{kA_1}{L} \Delta T$$

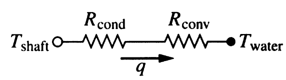
$$R_{\text{cond}} = \frac{\Delta T}{q} = \frac{L}{kA_1}$$

where A_1 is a cross sectional area and L is the distance from the inner core to the surface. The convection resistance is described by a similar equation, in which convective heat flow is described by the film coefficient of heat transfer, h:

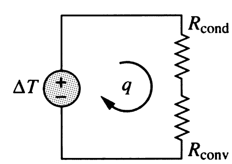
$$q = hA_2 \Delta T$$

$$R_{\text{conv}} = \frac{\Delta T}{q} = \frac{1}{hA_2}$$

where A_2 is the surface area of the shaft in contact with the water. The equivalent thermal resistance and the overall circuit model of the crankshaft quenching process are shown in the figures below.



Thermal resistance representation of quenching process



Electrical circuit representing the quenching process

EXAMPLE 3.2 Node Analysis

Problem

Solve for all unknown currents and voltages in the circuit of Figure 3.5.

Solution

Known Quantities: Source currents, resistor values.

Find: All node voltages and branch currents.

Schematics, Diagrams, Circuits, and Given Data: $I_1 = 10$ mA; $I_2 = 50$ mA; $R_1 = 1\ \text{k}\Omega$; $R_2 = 2\ \text{k}\Omega$; $R_3 = 10\ \text{k}\Omega$; $R_4 = 2\ \text{k}\Omega$.

Analysis: We follow the steps outlined in the Focus on Methodology box:

1. The reference (ground) node is chosen to be the node at the bottom of the circuit.
2. The circuit of Figure 3.5 is shown again in Figure 3.6, and two nodes are also shown in the figure. Thus, there are two independent variables in this circuit: v_1, v_2.

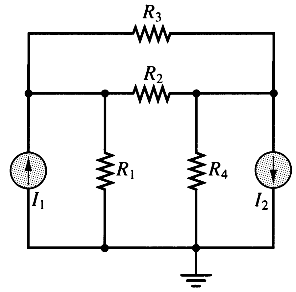


Figure 3.5

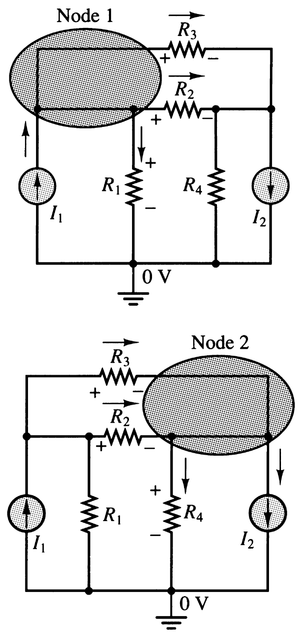


Figure 3.6

3. Applying KCL at nodes 1 and 2, we obtain

$$I_1 - \frac{v_1 - 0}{R_1} - \frac{v_1 - v_2}{R_2} - \frac{v_1 - v_2}{R_3} = 0 \qquad \text{node 1}$$

$$\frac{v_1 - v_2}{R_2} + \frac{v_1 - v_2}{R_3} - \frac{v_2 - 0}{R_4} - I_2 = 0 \qquad \text{node 2}$$

Now we can write the same equations more systematically as a function of the unknown node voltages, as was done in equation 3.9.

$$\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) v_1 + \left(-\frac{1}{R_2} - \frac{1}{R_3}\right) v_2 = I_1 \qquad \text{node 1}$$

$$\left(-\frac{1}{R_2} - \frac{1}{R_3}\right) v_1 + \left(\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right) v_2 = -I_2 \qquad \text{node 2}$$

4. We finally solve the system of equations. With some manipulation, the equations finally lead to the following form:

$$1.6v_1 - 0.6v_2 = 10$$
$$-0.6v_1 + 1.1v_2 = -50$$

These equations may be solved simultaneously to obtain

$$v_1 = -13.57 \text{ V}$$
$$v_2 = -52.86 \text{ V}$$

Knowing the node voltages, we can determine each of the branch currents and voltages in the circuit. For example, the current through the 10-kΩ resistor is given by

$$i_{10\text{ k}\Omega} = \frac{v_1 - v_2}{10{,}000} = 3.93 \text{ mA}$$

indicating that the initial (arbitrary) choice of direction for this current was the same as the actual direction of current flow. As another example, consider the current through the 1-kΩ resistor:

$$i_{1\text{ k}\Omega} = \frac{v_1}{1{,}000} = -13.57 \text{ mA}$$

In this case, the current is negative, indicating that current actually flows from ground to node 1, as it should, since the voltage at node 1 is negative with respect to ground. You may continue the branch-by-branch analysis started in this example to verify that the solution obtained in the example is indeed correct.

Comments: Note that we have chosen to assign a plus sign to currents entering a node and a minus sign to currents exiting a node; this choice is arbitrary (we could use the opposite convention), but we shall use it consistently in this book.

EXAMPLE 3.3 Node Analysis

LO1

Problem

Write the nodal equations and solve for the node voltages in the circuit of Figure 3.7.

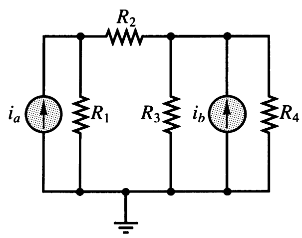


Figure 3.7

Solution

Known Quantities: Source currents, resistor values.

Find: All node voltages and branch currents.

Schematics, Diagrams, Circuits, and Given Data: $i_a = 1$ mA; $i_b = 2$ mA; $R_1 = 1$ kΩ; $R_2 = 500$ Ω; $R_3 = 2.2$ kΩ; $R_4 = 4.7$ kΩ.

Analysis: We follow the steps of the Focus on Methodology box.

1. The reference (ground) node is chosen to be the node at the bottom of the circuit.
2. See Figure 3.8. Two nodes remain after the selection of the reference node. Let us label these a and b and define voltages v_a and v_b. Both nodes are associated with independent variables.
3. We apply KCL at each of nodes a and b:

$$i_a - \frac{v_a}{R_1} - \frac{v_a - v_b}{R_2} = 0 \qquad \text{node } a$$

$$\frac{v_a - v_b}{R_2} + i_b - \frac{v_b}{R_3} - \frac{v_b}{R_4} = 0 \qquad \text{node } b$$

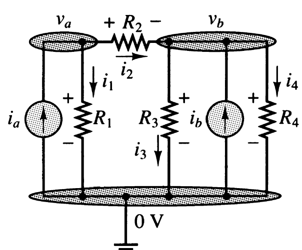


Figure 3.8

and rewrite the equations to obtain a linear system:

$$\left(\frac{1}{R_1}+\frac{1}{R_2}\right)v_a+\left(-\frac{1}{R_2}\right)v_b=i_a$$

$$\left(-\frac{1}{R_2}\right)v_a+\left(\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}\right)v_b=i_b$$

4. Substituting the numerical values in these equations, we get

$$3\times10^{-3}v_a-2\times10^{-3}v_b=1\times10^{-3}$$
$$-2\times10^{-3}v_a+2.67\times10^{-3}v_b=2\times10^{-3}$$

or

$$3v_a-2v_b=1$$
$$-2v_a+2.67v_b=2$$

The solution $v_a = 1.667$ V, $v_b = 2$ V may then be obtained by solving the system of equations.

LO1

EXAMPLE 3.4 Solution of Linear System of Equations Using Cramer's Rule

Problem

Solve the circuit equations obtained in Example 3.3, using Cramer's rule (see Appendix A).

Solution

Known Quantities: Linear system of equations.

Find: Node voltages.

Analysis: The system of equations generated in Example 3.3 may also be solved by using linear algebra methods, by recognizing that the system of equations can be written as

$$\begin{bmatrix}3 & -2\\ -2 & 2.67\end{bmatrix}\begin{bmatrix}v_a\\ v_b\end{bmatrix}=\begin{bmatrix}1\\ 2\end{bmatrix}$$

By using Cramer's rule (see Appendix A), the solution for the two unknown variables v_a and v_b can be written as follows:

$$v_a=\frac{\begin{vmatrix}1 & -2\\ 2 & 2.67\end{vmatrix}}{\begin{vmatrix}3 & -2\\ -2 & 2.67\end{vmatrix}}=\frac{(1)(2.67)-(-2)(2)}{(3)(2.67)-(-2)(-2)}=\frac{6.67}{4}=1.667\text{ V}$$

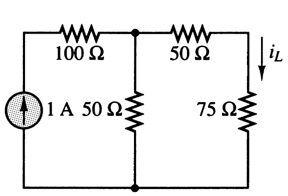
$$v_b=\frac{\begin{vmatrix}3 & 1\\ -2 & 2\end{vmatrix}}{\begin{vmatrix}3 & -2\\ -2 & 2.67\end{vmatrix}}=\frac{(3)(2)-(-2)(1)}{(3)(2.67)-(-2)(-2)}=\frac{8}{4}=2\text{ V}$$

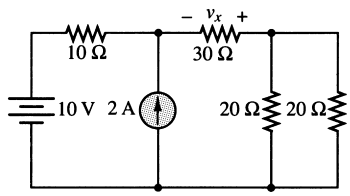
The result is the same as in Example 3.3.

Comments: While Cramer's rule is an efficient solution method for simple circuits (e.g., two nodes), it is customary to use computer-aided methods for larger circuits. Once the nodal equations have been set in the general form presented in equation 3.9, a variety of computer aids may be employed to compute the solution. You will find the solution to the same example computed using MathCad™ in the electronic files that accompany this book.

CHECK YOUR UNDERSTANDING

Find the current i_L in the circuit shown on the left, using the node voltage method.





Find the voltage v_x by the node voltage method for the circuit shown on the right.
Show that the answer to Example 3.3 is correct by applying KCL at one or more nodes.

Answers: 0.2857 A; −18 V

EXAMPLE 3.5

Problem

Use the node voltage analysis to determine the voltage v in the circuit of Figure 3.9. Assume that $R_1 = 2\ \Omega$, $R_2 = 1\ \Omega$, $R_3 = 4\ \Omega$, $R_4 = 3\ \Omega$, $I_1 = 2$ A, and $I_2 = 3$ A.

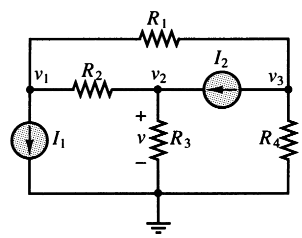


Figure 3.9 Circuit for Example 3.5

Solution

Known Quantities: Values of the resistors and the current sources.

Find: Voltage across R_3.

Analysis: Once again, we follow the steps outlined in the Focus on Methodology box.

1. The reference node is denoted in Figure 3.9.
2. Next, we define the three node voltages v_1, v_2, v_3, as shown in Figure 3.9.
3. Apply KCL at each of the $n - 1$ nodes, expressing each current in terms of the adjacent node voltages.

$$\frac{v_3 - v_1}{R_1} + \frac{v_2 - v_1}{R_2} - I_1 = 0 \qquad \text{node 1}$$

$$\frac{v_1 - v_2}{R_2} - \frac{v_2}{R_3} + I_2 = 0 \qquad \text{node 2}$$

$$\frac{v_1 - v_3}{R_1} - \frac{v_3}{R_4} - I_2 = 0 \qquad \text{node 3}$$

4. Solve the linear system of $n - 1 - m$ unknowns. Finally, we write the system of equations resulting from the application of KCL at the three nodes associated with independent