1

The Electromagnetic Model

1–1 Introduction

Stated in a simple fashion, ***electromagnetics*** is the study of the effects of electric charges at rest and in motion. From elementary physics we know that there are two kinds of charges: positive and negative. Both positive and negative charges are sources of an electric field. Moving charges produce a current, which gives rise to a magnetic field. Here we tentatively speak of electric field and magnetic field in a general way; more definitive meanings will be attached to these terms later. A ***field*** is a spatial distribution of a quantity, which may or may not be a function of time. A time-varying electric field is accompanied by a magnetic field, and vice versa. In other words, time-varying electric and magnetic fields are coupled, resulting in an electromagnetic field. Under certain conditions, time-dependent electromagnetic fields produce waves that radiate from the source.

The concept of fields and waves is essential in the explanation of action at a distance. For instance, we learned from elementary mechanics that masses attract each other. This is why objects fall toward the earth's surface. But since there are no elastic strings connecting a free-falling object and the earth, how do we explain this phenomenon? We explain this action-at-a-distance phenomenon by postulating the existence of a gravitational field. The possibilities of satellite communication and of receiving signals from space probes millions of miles away can be explained only by postulating the existence of electric and magnetic fields and electromagnetic waves. In this book, *Field and Wave Electromagnetics*, we study the principles and applications of the laws of electromagnetism that govern electromagnetic phenomena.

Electromagnetics is of fundamental importance to physicists and to electrical and computer engineers. Electromagnetic theory is indispensable in understanding the principle of atom smashers, cathode-ray oscilloscopes, radar, satellite communication, television reception, remote sensing, radio astronomy, microwave devices, optical fiber communication, transients in transmission lines, electromagnetic compatibility

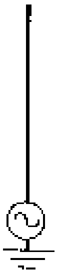


FIGURE 1–1
A monopole antenna.

problems, instrument-landing systems, electromechanical energy conversion, and so on. Circuit concepts represent a restricted version, a special case, of electromagnetic concepts. As we shall see in Chapter 7, when the source frequency is very low so that the dimensions of a conducting network are much smaller than the wavelength, we have a quasi-static situation, which simplifies an electromagnetic problem to a circuit problem. However, we hasten to add that circuit theory is itself a highly developed, sophisticated discipline. It applies to a different class of electrical engineering problems, and it is important in its own right.

Two situations illustrate the inadequacy of circuit-theory concepts and the need for electromagnetic-field concepts. Figure 1–1 depicts a monopole antenna of the type we see on a walkie-talkie. On transmit, the source at the base feeds the antenna with a message-carrying current at an appropriate carrier frequency. From a circuit-theory point of view, the source feeds into an open circuit because the upper tip of the antenna is not connected to anything physically; hence no current would flow, and nothing would happen. This viewpoint, of course, cannot explain why communication can be established between walkie-talkies at a distance. Electromagnetic concepts must be used. We shall see in Chapter 11 that when the length of the antenna is an appreciable part of the carrier wavelength,† a nonuniform current will flow along the open-ended antenna. This current radiates a time-varying electromagnetic field in space, which propagates as an electromagnetic wave and induces currents in other antennas at a distance.

In Fig. 1–2 we show a situation in which an electromagnetic wave is incident from the left on a large conducting wall containing a small hole (aperture). Electromagnetic fields will exist on the right side of the wall at points, such as P in the figure, that are not necessarily directly behind the aperture. Circuit theory is obviously inadequate here for the determination (or even the explanation of the existence) of the field at P. The situation in Fig. 1–2, however, represents a problem of practical importance as its solution is relevant in evaluating the shielding effectiveness of the conducting wall.

† The product of the wavelength and the frequency of an a-c source is the velocity of wave propagation.

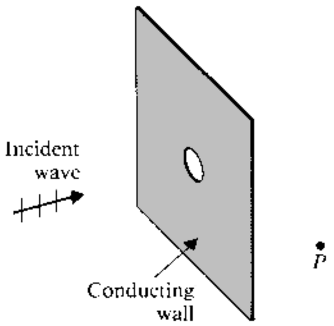


FIGURE 1–2
An electromagnetic problem.

Generally speaking, circuit theory deals with lumped-parameter systems—circuits consisting of components characterized by lumped parameters such as resistances, inductances, and capacitances. Voltages and currents are the main system variables. For d-c circuits the system variables are constants, and the governing equations are algebraic equations. The system variables in a-c circuits are time-dependent; they are scalar quantities and are independent of space coordinates. The governing equations are ordinary differential equations. On the other hand, most electromagnetic variables are functions of time as well as of space coordinates. Many are vectors with both a magnitude and a direction, and their representation and manipulation require a knowledge of vector algebra and vector calculus. Even in static cases the governing equations are, in general, partial differential equations. It is essential that we be equipped to handle vector quantities and variables that are both time- and space-dependent. The fundamentals of vector algebra and vector calculus will be developed in Chapter 2. Techniques for solving partial differential equations are needed in dealing with certain types of electromagnetic problems. These techniques will be discussed in Chapter 4. The importance of acquiring a facility in the use of these mathematical tools in the study of electromagnetics cannot be overemphasized.

Students who have mastered circuit theory may initially have the impression that electromagnetic theory is abstract. In fact, electromagnetic theory is no more abstract than circuit theory in the sense that the validity of both can be verified by experimentally measured results. In electromagnetics there is a need to define more quantities and to use more mathematical manipulations in order to develop a logical and complete theory that can explain a much wider variety of phenomena. The challenge of field and wave electromagnetics is not in the abstractness of the subject matter but rather in the process of mastering the electromagnetic model and the associated rules of operation. Dedication to acquiring this mastery will help us to meet the challenge and reap immeasurable satisfaction.

1–2 The Electromagnetic Model

There are two approaches in the development of a scientific subject: the inductive approach and the deductive approach. Using the inductive approach, one follows the historical development of the subject, starting with the observations of some simple experiments and inferring from them laws and theorems. It is a process of reasoning from particular phenomena to general principles. The deductive approach, on the other hand, postulates a few fundamental relations for an idealized model. The postulated relations are axioms, from which particular laws and theorems can be derived. The validity of the model and the axioms is verified by their ability to predict consequences that check with experimental observations. In this book we prefer to use the deductive or axiomatic approach because it is more elegant and enables the development of the subject of electromagnetics in an orderly way.

The idealized model we adopt for studying a scientific subject must relate to real-world situations and be able to explain physical phenomena; otherwise, we would be engaged in mental exercises for no purpose. For example, a theoretical model could be built, from which one might obtain many mathematical relations; but, if these relations disagreed with observed results, the model would be of no use. The mathematics might be correct, but the underlying assumptions of the model could be wrong, or the implied approximations might not be justified.

Three essential steps are involved in building a theory on an idealized model. *First*, some basic quantities germane to the subject of study are defined. *Second*, the rules of operation (the mathematics) of these quantities are specified. *Third*, some fundamental relations are postulated. These postulates or laws are invariably based on numerous experimental observations acquired under controlled conditions and synthesized by brilliant minds. A familiar example is the circuit theory built on a circuit model of ideal sources and pure resistances, inductances, and capacitances. In this case the basic quantities are voltages (V), currents (I), resistances (R), inductances (L), and capacitances (C); the rules of operations are those of algebra, ordinary differential equations, and Laplace transformation; and the fundamental postulates are Kirchhoff's voltage and current laws. Many relations and formulas can be derived from this basically rather simple model, and the responses of very elaborate networks can be determined. The validity and value of the model have been amply demonstrated.

In a like manner, an electromagnetic theory can be built on a suitably chosen electromagnetic model. In this section we shall take the first step of defining the basic quantities of electromagnetics. The second step, the rules of operation, encompasses vector algebra, vector calculus, and partial differential equations. The fundamentals of vector algebra and vector calculus will be discussed in Chapter 2 (Vector Analysis), and the techniques for solving partial differential equations will be introduced when these equations arise later in the book. The third step, the fundamental postulates, will be presented in three substeps in Chapters 3, 6, and 7 as we deal with static electric fields, steady magnetic fields, and electromagnetic fields, respectively.

The quantities in our electromagnetic model can be divided roughly into two categories: source quantities and field quantities. The source of an electromagnetic field is invariably electric charges at rest or in motion. However, an electromagnetic field may cause a redistribution of charges, which will, in turn, change the field; hence the separation between the cause and the effect is not always so distinct.

We use the symbol q (sometimes Q) to denote ***electric charge***. Electric charge is a fundamental property of matter and exists only in positive or negative integral multiples of the charge on an electron, $-e$.[†]

$$\boxed{e = 1.60 \times 10^{-19} \quad \text{(C)},} \tag{1–1}$$

where C is the abbreviation of the unit of charge, coulomb.[‡] It is named after the French physicist Charles A. de Coulomb, who formulated Coulomb's law in 1785. (Coulomb's law will be discussed in Chapter 3.) A coulomb is a very large unit for electric charge; it takes $1/(1.60 \times 10^{-19})$ or 6.25 million trillion electrons to make up -1 C. In fact, two 1 C charges 1 m apart will exert a force of approximately 1 million tons on each other. Some other physical constants for the electron are listed in Appendix B–2.

The principle of ***conservation of electric charge***, like the principle of conservation of momentum, is a fundamental postulate or law of physics. It states that electric charge is conserved; that is, it can neither be created nor be destroyed. This is a law of nature and cannot be derived from other principles or relations. Its truth has never been questioned or doubted in practice.

Electric charges can move from one place to another and can be redistributed under the influence of an electromagnetic field; but the algebraic sum of the positive and negative charges in a closed (isolated) system remains unchanged. ***The principle of conservation of electric charge must be satisfied at all times and under any circumstances.*** It is represented mathematically by the ***equation of continuity***, which we will discuss in Section 5–4. Any formulation or solution of an electromagnetic problem that violates the principle of conservation of electric charge ***must be*** incorrect. We recall that the Kirchhoff's current law in circuit theory, which maintains that the sum of all the currents leaving a junction must equal the sum of all the currents entering the junction, is an assertion of the conservation property of electric charge. (Implicit in the current law is the assumption that there is no cumulation of charge at the junction.)

Although, in a microscopic sense, electric charge either does or does not exist at a point in a discrete manner, these abrupt variations on an atomic scale are unimportant when we consider the electromagnetic effects of large aggregates of charges. In constructing a macroscopic or large-scale theory of electromagnetism we find that the use of smoothed-out average density functions yields very good results. (The same approach is used in mechanics where a smoothed-out mass density function is defined, in spite of the fact that mass is associated only with elementary particles in a discrete

---

[†] In 1962, Murray Gell-Mann hypothesized ***quarks*** as the basic building blocks of matter. Quarks were predicted to carry a fraction of the charge of an electron, and their existence has since been verified experimentally.

[‡] The system of units will be discussed in Section 1–3.

manner on an atomic scale.) We define a ***volume charge density***, ρ, as a source quantity as follows:

$$\rho = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta v} \qquad (\text{C/m}^3), \tag{1-2}$$

where Δq is the amount of charge in a very small volume Δv. How small should Δv be? It should be small enough to represent an accurate variation of ρ but large enough to contain a very large number of discrete charges. For example, an elemental cube with sides as small as 1 micron (10^{-6} m or 1 μm) has a volume of 10^{-18} m^3, which will still contain about 10^{11} (100 billion) atoms. A smoothed-out function of space coordinates, ρ, defined with such a small Δv is expected to yield accurate macroscopic results for nearly all practical purposes.

In some physical situations an amount of charge Δq may be identified with an element of surface Δs or an element of line $\Delta \ell$. In such cases it will be more appropriate to define a ***surface charge density***, ρ_s, or a ***line charge density***, ρ_ℓ:

$$\rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} \qquad (\text{C/m}^2), \tag{1-3}$$

$$\rho_\ell = \lim_{\Delta \ell \to 0} \frac{\Delta q}{\Delta \ell} \qquad (\text{C/m}). \tag{1-4}$$

Except for certain special situations, charge densities vary from point to point; hence ρ, ρ_s, and ρ_ℓ are, in general, point functions of space coordinates.

Current is the rate of change of charge with respect to time; that is,

$$I = \frac{dq}{dt} \qquad (\text{C/s or A}), \tag{1-5}$$

where I itself may be time-dependent. The unit of current is coulomb per second (C/s), which is the same as ampere (A). A current must flow through a finite area (a conducting wire of a finite cross section, for instance); hence it is not a point function. In electromagnetics we define a vector point function ***volume current density*** (or simply ***current density***) $\mathbf{J}$, which measures the amount of current flowing through a unit area normal to the direction of current flow. The boldfaced $\mathbf{J}$ is a vector whose magnitude is the current per unit area (A/m^2) and whose direction is the direction of current flow. We shall elaborate on the relation between I and $\mathbf{J}$ in Chapter 5. For very good conductors, high-frequency alternating currents are confined in the surface layer as a current sheet, instead of flowing throughout the interior of the conductor. In such cases there is a need to define a ***surface current density*** $\mathbf{J}_s$, which is the current per unit width on the conductor surface normal to the direction of current flow and has the unit of ampere per meter (A/m).

There are four fundamental *vector* field quantities in electromagnetics: ***electric field intensity*** $\mathbf{E}$, ***electric flux density*** (or ***electric displacement***) $\mathbf{D}$, ***magnetic flux

TABLE 1–1
Fundamental Electromagnetic Field Quantities

Symbols and Units for Field Quantities	Field Quantity	Symbol	Unit
Electric	Electric field intensity	**E**	V/m
	Electric flux density (Electric displacement)	**D**	C/m^2
Magnetic	Magnetic flux density	**B**	T
	Magnetic field intensity	**H**	A/m

density **B**, and ***magnetic field intensity*** **H**. The definition and physical significance of these quantities will be explained fully when they are introduced later in the book. At this time we want only to establish the following. Electric field intensity **E** is the only vector needed in discussing electrostatics (effects of stationary electric charges) in free space; it is defined as the electric force on a unit test charge. Electric displacement vector **D** is useful in the study of electric field in material media, as we shall see in Chapter 3. Similarly, magnetic flux density **B** is the only vector needed in discussing magnetostatics (effects of steady electric currents) in free space and is related to the magnetic force acting on a charge moving with a given velocity. The magnetic field intensity vector **H** is useful in the study of magnetic field in material media. The definition and significance of **B** and **H** will be discussed in Chapter 6.

The four fundamental electromagnetic field quantities, together with their units, are tabulated in Table 1–1. In Table 1–1, V/m is volt per meter, and T stands for tesla or volt-second per square meter. When there is no time variation (as in static, steady, or stationary cases), the electric field quantities **E** and **D** and the magnetic field quantities **B** and **H** form two separate vector pairs. In time-dependent cases, however, electric and magnetic field quantities are coupled; that is, time-varying **E** and **D** will give rise to **B** and **H**, and vice versa. All four quantities are point functions; they are defined at every point in space and, in general, are functions of space coordinates. Material (or medium) properties determine the relations between **E** and **D** and between **B** and **H**. These relations are called the ***constitutive relations*** of a medium and will be examined later.

The principal objective of studying electromagnetism is to understand the interaction between charges and currents at a distance based on the electromagnetic model. Fields and waves (time- and space-dependent fields) are basic conceptual quantities of this model. Fundamental postulates will relate **E**, **D**, **B**, **H**, and the source quantities; and derived relations will lead to the explanation and prediction of electromagnetic phenomena.

TABLE 1–2
Fundamental SI Units

Quantity	Unit	Abbreviation
Length	meter	m
Mass	kilogram	kg
Time	second	s
Current	ampere	A

1–3 SI Units and Universal Constants

A measurement of any physical quantity must be expressed as a number followed by a unit. Thus we may talk about a length of three meters, a mass of two kilograms, and a time period of ten seconds. To be useful, a unit system should be based on some fundamental units of convenient (practical) sizes. In mechanics, all quantities can be expressed in terms of three basic units (for length, mass, and time). In electromagnetics a fourth basic unit (for current) is needed. The SI (***International System of Units*** or ***Le Système International d'Unités***) is an ***MKSA system*** built from the four fundamental units listed in Table 1–2. All other units used in electromagnetics, including those appearing in Table 1–1, are derived units expressible in terms of meters, kilograms, seconds, and amperes. For example, the unit for charge, coulomb (C), is ampere-second (A·s); the unit for electric field intensity (V/m) is $kg{\cdot}m/A{\cdot}s^3$; and the unit for magnetic flux density, tesla (T), is $kg/A{\cdot}s^2$. More complete tables of the units for various quantities are given in Appendix A.

The official SI definitions, as adopted by the International Committee on Weights and Measures, are as follows:†

Meter. Once the length between two scratches on a platinum-iridium bar (and originally calculated as one ten-millionth of the distance between the North Pole and the equator through Paris, France), is now defined by reference to the *second* (see below) and the speed of light, which in a vacuum is 299,792,458 meters per second.

Kilogram. Mass of a standard bar made of a platinum-iridium alloy and kept inside a set of nested enclosures that protect it from contamination and mishandling. It rests at the International Bureau of Weights and Measures in Sèvres, outside Paris.

Second. 9,192,631,770 periods of the electromagnetic radiation emitted by a particular transition of a cesium atom.

† P. Wallich, "Volts and amps are not what they used to be," *IEEE Spectrum*, vol. 24, pp. 44–49, March 1987.

Ampere. The constant current that, if maintained in two straight parallel conductors of infinite length and negligible circular cross section, and placed one meter apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} newton per meter of length. (A newton is the force that gives a mass of one kilogram an acceleration of one meter per second squared.)

In our electromagnetic model there are three universal constants, in addition to the field quantities listed in Table 1–1. They relate to the properties of the free space (vacuum). They are as follows: ***velocity of electromagnetic wave*** (including light) in free space, c; ***permittivity*** of free space, ϵ_0; and ***permeability*** of free space, μ_0. Many experiments have been performed for precise measurement of the velocity of light, to many decimal places. For our purpose it is sufficient to remember that

$$\boxed{c \cong 3 \times 10^8 \quad \text{(m/s).}} \quad \text{(in free space)} \tag{1–6}$$

The other two constants, ϵ_0 and μ_0, pertain to electric and magnetic phenomena, respectively: ϵ_0 is the proportionality constant between the electric flux density **D** and the electric field intensity **E** in free space, such that

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E};} \quad \text{(in free space)} \tag{1–7}$$

μ_0 is the proportionality constant between the magnetic flux density **B** and the magnetic field intensity **H** in free space, such that

$$\boxed{\mathbf{H} = \frac{1}{\mu_0} \mathbf{B}.} \quad \text{(in free space)} \tag{1–8}$$

The values of ϵ_0 and μ_0 are determined by the choice of the unit system, and they are not independent. In the ***SI system*** (rationalized† MKSA system), which is almost universally adopted for electromagnetics work, the permeability of free space is chosen to be

$$\boxed{\mu_0 = 4\pi \times 10^{-7} \quad \text{(H/m).}} \quad \text{(in free space)} \tag{1–9}$$

where H/m stands for henry per meter. With the values of c and μ_0 fixed in Eqs. (1–6) and (1–9) the value of the permittivity of free space is then derived from the following

† This system of units is said to be ***rationalized*** because the factor 4π does not appear in the Maxwell's equations (the fundamental postulates of electromagnetism). This factor, however, will appear in many derived relations. In the unrationalized MKSA system, μ_0 would be 10^{-7} (H/m), and the factor 4π would appear in the Maxwell's equations.

TABLE 1-3
Universal Constants in SI Units

Universal Constants	Symbol	Value	Unit
Velocity of light in free space	c	3×10^8	m/s
Permeability of free space	μ_0	$4\pi \times 10^{-7}$	H/m
Permittivity of free space	ϵ_0	$\frac{1}{36\pi} \times 10^{-9}$	F/m

relationships:

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \quad \text{(m/s)} \tag{1-10}$$

or

$$\begin{aligned} \epsilon_0 = \frac{1}{c^2 \mu_0} &\cong \frac{1}{36\pi} \times 10^{-9} \\ &\cong 8.854 \times 10^{-12} \quad \text{(F/m)}, \end{aligned} \tag{1-11}$$

where F/m is the abbreviation for farad per meter. The three universal constants and their values are summarized in Table 1-3.

Now that we have defined the basic quantities and the universal constants of the electromagnetic model, we can develop the various subjects in electromagnetics. But, before we do that, we must be equipped with the appropriate mathematical tools. In the following chapter we discuss the basic rules of operation for vector algebra and vector calculus.

Review Questions

R.1–1 What is electromagnetics?

R.1–2 Describe two phenomena or situations, other than those depicted in Figs. 1–1 and 1–2, that cannot be adequately explained by circuit theory.

R.1–3 What are the three essential steps in building an idealized model for the study of a scientific subject?

R.1–4 What are the four fundamental SI units in electromagnetics?

R.1–5 What are the four fundamental field quantities in the electromagnetic model? What are their units?

R.1–6 What are the three universal constants in the electromagnetic model, and what are their relations?

R.1–7 What are the source quantities in the electromagnetic model?

2

Vector Analysis

2–1 Introduction

As we noted in Chapter 1, some of the quantities in electromagnetics (such as charge, current, and energy) are scalars; and some others (such as electric and magnetic field intensities) are vectors. Both scalars and vectors can be functions of time and position. At a given time and position, a ***scalar*** is completely specified by its magnitude (positive or negative, together with its unit). Thus we can specify, for instance, a charge of -1 μC at a certain location at $t = 0$. The specification of a ***vector*** at a given location and time, on the other hand, requires both a magnitude and a direction. How do we specify the direction of a vector? In a three-dimensional space, three numbers are needed, and these numbers depend on the choice of a coordinate system. Conversion of a given vector from one coordinate system to another will change these numbers. However, physical laws and theorems relating various scalar and vector quantities certainly must hold irrespective of the coordinate system. The general expressions of the laws of electromagnetism, therefore, do not require the specification of a coordinate system. A particular coordinate system is chosen only when a problem of a given geometry is to be analyzed. For example, if we are to determine the magnetic field at the center of a current-carrying wire loop, it is more convenient to use rectangular coordinates if the loop is rectangular, whereas polar coordinates (two-dimensional) will be more appropriate if the loop is circular in shape. The basic electromagnetic relation governing the solution of such a problem is the same for both geometries.

Three main topics will be dealt with in this chapter on vector analysis:

1. Vector algebra—addition, subtraction, and multiplication of vectors.
2. Orthogonal coordinate systems—Cartesian, cylindrical, and spherical coordinates.
3. Vector calculus—differentiation and integration of vectors; line, surface, and volume integrals; "del" operator; gradient, divergence, and curl operations.

Throughout the rest of this book we will decompose, combine, differentiate, integrate, and otherwise manipulate vectors. It is *imperative* to acquire a facility in vector algebra and vector calculus. In a three-dimensional space a vector relation is, in fact, three scalar relations. The use of vector-analysis techniques in electromagnetics leads to concise and elegant formulations. A deficiency in vector analysis in the study of electromagnetics is similar to a deficiency in algebra and calculus in the study of physics; and it is obvious that these deficiencies cannot yield fruitful results.

In solving practical problems we always deal with regions or objects of a given shape, and it is necessary to express general formulas in a coordinate system appropriate for the given geometry. For example, the familiar rectangular (x, y, z) coordinates are, obviously, awkward to use for problems involving a circular cylinder or a sphere because the boundaries of a circular cylinder and a sphere cannot be described by constant values of x, y, and z. In this chapter we discuss the three most commonly used orthogonal (perpendicular) coordinate systems and the representation and operation of vectors in these systems. Familiarity with these coordinate systems is essential in the solution of electromagnetic problems.

Vector calculus pertains to the differentiation and integration of vectors. By defining certain differential operators we can express the basic laws of electromagnetism in a concise way that is invariant with the choice of a coordinate system. In this chapter we introduce the techniques for evaluating different types of integrals involving vectors, and we define and discuss the various kinds of differential operators.

2–2 Vector Addition and Subtraction

We know that a vector has a magnitude and a direction. A vector **A** can be written as

$$\mathbf{A} = \mathbf{a}_A A, \tag{2-1}$$

where A is the magnitude (and has the unit and dimension) of **A**,

$$A = |\mathbf{A}|, \tag{2-2}$$

and $\mathbf{a}_A$ is a dimensionless unit vector* with a unity magnitude having the direction of **A**. Thus,

$$\mathbf{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{A}}{A}. \tag{2-3}$$

The vector **A** can be represented graphically by a directed straight-line segment of a length $|\mathbf{A}| = A$ with its arrowhead pointing in the direction of $\mathbf{a}_A$, as shown in Fig. 2–1. Two vectors are equal if they have the same magnitude and the same direction, even

* In some books the unit vector in the direction of **A** is variously denoted by $\hat{\mathbf{A}}$, $\mathbf{u}_A$, or $\mathbf{i}_A$. We prefer to write **A** as in Eq. (2–1) instead of as $\mathbf{A} = \hat{\mathbf{A}}A$. A vector going from point P_1 to point P_2 will then be written as $\mathbf{a}_{P_1P_2}(\overline{P_1P_2})$ instead of as $\widehat{\mathbf{P}_1\mathbf{P}_2}(P_1P_2)$, which is somewhat cumbersome. The symbols **u** and **i** are used for velocity and current, respectively.

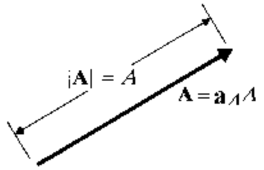


FIGURE 2–1
Graphical representation of vector **A**.

though they may be displaced in space. Since it is difficult to write boldfaced letters by hand, it is a common practice to use an arrow or a bar over a letter ($\vec{A}$ or $\bar{A}$) or a wiggly line under a letter ($\underset{\sim}{A}$) to distinguish a vector from a scalar. This distinguishing mark, once chosen, *should never be omitted* whenever and wherever vectors are written.

Two vectors **A** and **B**, which are not in the same direction nor in opposite directions, such as given in Fig. 2–2(a), determine a plane. Their sum is another vector **C** in the same plane. $\mathbf{C} = \mathbf{A} + \mathbf{B}$ can be obtained graphically in two ways.

1. By the parallelogram rule: The resultant **C** is the diagonal vector of the parallelogram formed by **A** and **B** drawn from the same point, as shown in Fig. 2–2(b).
2. By the head-to-tail rule: The head of **A** connects to the tail of **B**. Their sum **C** is the vector drawn from the tail of **A** to the head of **B**; and vectors **A**, **B**, and **C** form a triangle, as shown in Fig. 2–2(c).

It is obvious that vector addition obeys the commutative and associative laws.

$$\text{Commutative law:} \quad \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}. \tag{2-4}$$

$$\text{Associative law:} \quad \mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}. \tag{2-5}$$

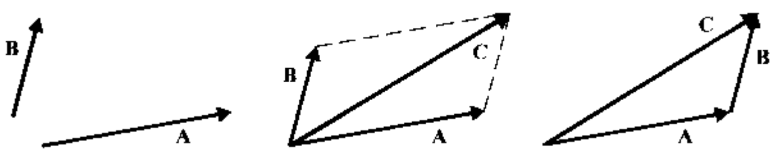
Vector subtraction can be defined in terms of vector addition in the following way:

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}), \tag{2-6}$$

where $-\mathbf{B}$ is the negative of vector **B**; that is, $-\mathbf{B}$ has the same magnitude as **B**, but its direction is opposite to that of **B**. Thus

$$-\mathbf{B} = (-\mathbf{a}_B)B. \tag{2-7}$$

The operation represented by Eq. (2-6) is illustrated in Fig. 2–3.



(a) Two vectors, **A** and **B**. (b) Parallelogram rule. (c) Head-to-tail rule.

FIGURE 2–2
Vector addition, $\mathbf{C} = \mathbf{A} + \mathbf{B}$.

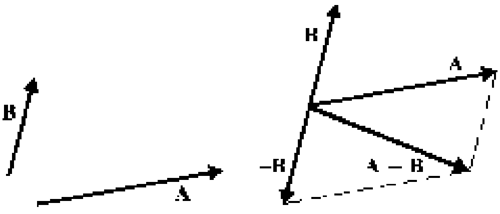


FIGURE 2–3
Vector subtraction.

2–3 Products of Vectors

Multiplication of a vector **A** by a positive scalar k changes the magnitude of **A** by k times without changing its direction (k can be either greater or less than 1).

$$k\mathbf{A} = \mathbf{a}_A(kA). \tag{2–8}$$

It is not sufficient to say "the multiplication of one vector by another" or "the product of two vectors" because there are two distinct and very different types of products of two vectors. They are (1) scalar or dot products, and (2) vector or cross products. These will be defined in the following subsections.

2–3.1 SCALAR OR DOT PRODUCT

The scalar or dot product of two vectors **A** and **B**, denoted by **A · B**, is a scalar, which equals the product of the magnitudes of **A** and **B** and the cosine of the angle between them. Thus,

$$\boxed{\mathbf{A} \cdot \mathbf{B} \triangleq AB \cos \theta_{AB}.} \tag{2–9}$$

In Eq. (2–9) the symbol $\triangleq$ signifies "equal by definition," and θ_{AB} is the *smaller* angle between **A** and **B** and is less than π radians (180°), as indicated in Fig. 2–4. The dot product of two vectors (1) is less than or equal to the product of their magnitudes; (2) can be either a positive or a negative quantity, depending on whether the angle between them is smaller or larger than $\pi/2$ radians (90°); (3) is equal to the product

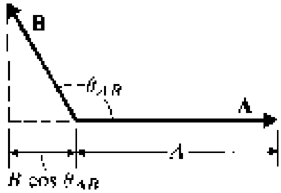


FIGURE 2–4
Illustrating the dot product of **A** and **B**.

of the magnitude of one vector and the projection of the other vector upon the first one; and (4) is zero when the vectors are perpendicular to each other. It is evident that

$$\mathbf{A} \cdot \mathbf{A} = A^2 \tag{2–10}$$

or

$$A = \sqrt{\mathbf{A} \cdot \mathbf{A}}. \tag{2–11}$$

Equation (2–11) enables us to find the magnitude of a vector when the expression of the vector is given in any coordinate system.

The dot product is commutative and distributive.

$$\text{Commutative law:} \quad \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}. \tag{2–12}$$

$$\text{Distributive law:} \quad \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}. \tag{2–13}$$

The commutative law is obvious from the definition of the dot product in Eq. (2–9), and the proof of Eq. (2–13) is left as an exercise. The associative law does not apply to the dot product, since no more than two vectors can be so multiplied and an expression such as $\mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C}$ is meaningless.

EXAMPLE 2–1 Prove the law of cosines for a triangle.

Solution The law of cosines is a scalar relationship that expresses the length of a side of a triangle in terms of the lengths of the two other sides and the angle between them. Referring to Fig. 2–5, we find the law of cosines states that

$$C = \sqrt{A^2 + B^2 - 2AB \cos \alpha}.$$

We prove this by considering the sides as vectors; that is,

$$\mathbf{C} = \mathbf{A} + \mathbf{B}.$$

Taking the dot product of $\mathbf{C}$ with itself, we have, from Eqs. (2–10) and (2–13),

$$\begin{aligned} C^2 = \mathbf{C} \cdot \mathbf{C} &= (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} + \mathbf{B}) \\ &= \mathbf{A} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{B} + 2\mathbf{A} \cdot \mathbf{B} \\ &= A^2 + B^2 + 2AB \cos \theta_{AB}. \end{aligned}$$

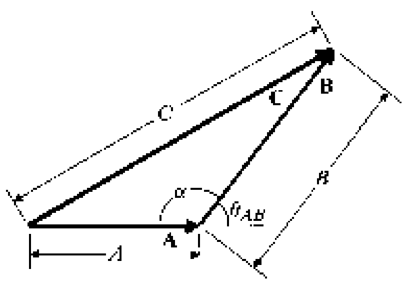


FIGURE 2–5
Illustrating Example 2–1.

Note that θ_{AB} is, by definition, the *smaller* angle between **A** and **B** and is equal to $(180° - \alpha)$; hence $\cos\theta_{AB} = \cos(180° - \alpha) = -\cos\alpha$. Therefore,

$$C^2 = A^2 + B^2 - 2AB\cos\alpha,$$

and the law of cosines follows directly. ∎

2–3.2 VECTOR OR CROSS PRODUCT

The vector or cross product of two vectors **A** and **B**, denoted by $\mathbf{A} \times \mathbf{B}$, is a vector perpendicular to the plane containing **A** and **B**; its magnitude is $AB\sin\theta_{AB}$, where θ_{AB} is the *smaller* angle between **A** and **B**, and its direction follows that of the thumb of the right hand when the fingers rotate from **A** to **B** through the angle θ_{AB} (the right-hand rule).

$$\boxed{\mathbf{A} \times \mathbf{B} \triangleq \mathbf{a}_n |AB\sin\theta_{AB}|.} \tag{2–14}$$

This is illustrated in Fig. 2–6. Since $B\sin\theta_{AB}$ is the height of the parallelogram formed by the vectors **A** and **B**, we recognize that the magnitude of $\mathbf{A} \times \mathbf{B}$, $|AB\sin\theta_{AB}|$, which is always positive, is numerically equal to the area of the parallelogram.

Using the definition in Eq. (2–14) and following the right-hand rule, we find that

$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}. \tag{2–15}$$

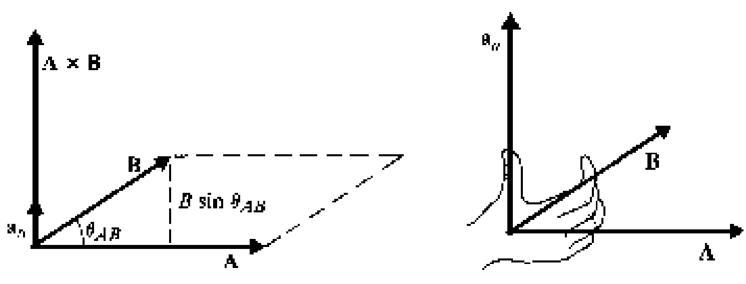
Hence the cross product is *not* commutative. We can see that the cross product obeys the distributive law,

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}. \tag{2–16}$$

Can you show this in general without resolving the vectors into rectangular components?

The vector product is obviously *not* associative; that is,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}. \tag{2–17}$$



(a) $\mathbf{A} \times \mathbf{B} = \mathbf{a}_n |AB\sin\theta_{AB}|$. (b) The right-hand rule.

FIGURE 2–6
Cross product of **A** and **B**, $\mathbf{A} \times \mathbf{B}$.

The vector representing the triple product on the left side of the expression above is perpendicular to **A** and lies in the plane formed by **B** and **C**, whereas that on the right side is perpendicular to **C** and lies in the plane formed by **A** and **B**. The order in which the two vector products are performed is therefore vital, and *in no case should the parentheses be omitted.*

EXAMPLE 2–2 The motion of a rigid disk rotating about its axis shown in Fig. 2–7(a) can be described by an angular velocity vector $\boldsymbol{\omega}$. The direction of $\boldsymbol{\omega}$ is along the axis and follows the right-hand rule; that is, if the fingers of the right hand bend in the direction of rotation, the thumb points to the direction of $\boldsymbol{\omega}$. Find the vector expression for the lineal velocity of a point on the disk, which is at a distance d from the axis of rotation.

Solution From mechanics we know that the magnitude of the lineal velocity, $\mathbf{v}$, of a point P at a distance d from the rotating axis is ωd and the direction is always tangential to the circle of rotation. However, since the point P is moving, the direction of $\mathbf{v}$ changes with the position of P. How do we write its vector representation?

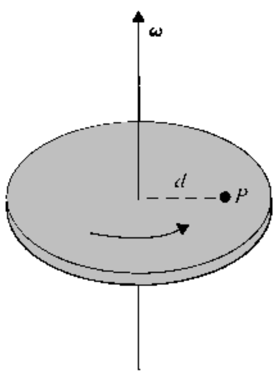
Let O be the origin of the chosen coordinate system. The position vector of the point P can be written as $\mathbf{R}$, as shown in Fig. 2–7(b). We have

$$|\mathbf{v}| = \omega d = \omega R \sin\theta.$$

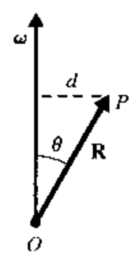
No matter where the point P is, the direction of $\mathbf{v}$ is always perpendicular to the plane containing the vectors $\boldsymbol{\omega}$ and $\mathbf{R}$. Hence we can write, very simply,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{R},$$

which represents correctly both the magnitude and the direction of the lineal velocity of P.



(a) A rotating disk.



(b) Vector representation.

FIGURE 2–7
Illustrating Example 2–2.

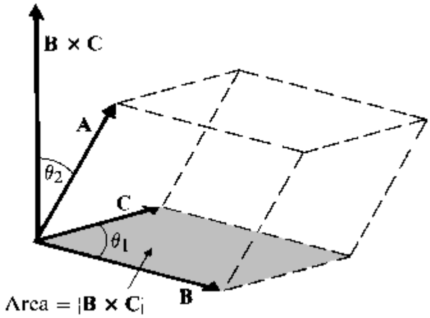


FIGURE 2–8
Illustrating scalar triple product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$.

2–3.3 PRODUCT OF THREE VECTORS

There are two kinds of products of three vectors; namely, the ***scalar triple product*** and the ***vector triple product***. The scalar triple product is much the simpler of the two and has the following property:

$$\boxed{\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}).} \tag{2–18}$$

Note the cyclic permutation of the order of the three vectors **A**, **B**, and **C**. Of course,

$$\begin{aligned} \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= -\mathbf{A} \cdot (\mathbf{C} \times \mathbf{B}) \\ &= -\mathbf{B} \cdot (\mathbf{A} \times \mathbf{C}) \\ &= -\mathbf{C} \cdot (\mathbf{B} \times \mathbf{A}). \end{aligned} \tag{2–19}$$

As can be seen from Fig. 2–8, each of the three expressions in Eq. (2–18) has a magnitude equal to the volume of the parallelepiped formed by the three vectors **A**, **B**, and **C**. The parallelepiped has a base with an area equal to $|\mathbf{B} \times \mathbf{C}| = |BC \sin \theta_1|$ and a height equal to $|A \cos \theta_2|$; hence the volume is $|ABC \sin \theta_1 \cos \theta_2|$.

The vector triple product $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ can be expanded as the difference of two simple vectors as follows:

$$\boxed{\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}).} \tag{2–20}$$

Equation (2–20) is known as the "***back-cab*" *rule*** and is a useful vector identity. (Note "BAC-CAB" on the right side of the equation!)

EXAMPLE 2–3[†] Prove the back-cab rule of vector triple product.

[†] The back-cab rule can be verified in a straightforward manner by expanding the vectors in the Cartesian coordinate system (Problem P.2–12). Only those interested in a general proof need to study this example.

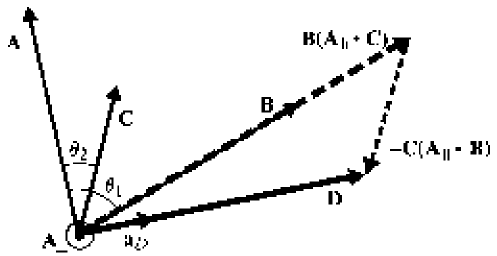


FIGURE 2–9
Illustrating the back-cab rule of vector triple product.

Solution In order to prove Eq. (2–20) it is convenient to expand $\mathbf{A}$ into two components:

$$\mathbf{A} = \mathbf{A}_{\parallel} + \mathbf{A}_{\perp},$$

where $\mathbf{A}_{\parallel}$ and $\mathbf{A}_{\perp}$ are parallel and perpendicular, respectively, to the plane containing $\mathbf{B}$ and $\mathbf{C}$. Because the vector representing $(\mathbf{B} \times \mathbf{C})$ is also perpendicular to the plane, the cross product of $\mathbf{A}_{\perp}$ and $(\mathbf{B} \times \mathbf{C})$ vanishes. Let $\mathbf{D} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$. Since only $\mathbf{A}_{\parallel}$ is effective here, we have

$$\mathbf{D} = \mathbf{A}_{\parallel} \times (\mathbf{B} \times \mathbf{C}).$$

Referring to Fig. 2–9, which shows the plane containing $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{A}_{\parallel}$, we note that $\mathbf{D}$ lies in the same plane and is normal to $\mathbf{A}_{\parallel}$. The magnitude of $(\mathbf{B} \times \mathbf{C})$ is $BC \sin(\theta_1 - \theta_2)$, and that of $\mathbf{A}_{\parallel} \times (\mathbf{B} \times \mathbf{C})$ is $A_{\parallel}BC \sin(\theta_1 - \theta_2)$. Hence,

$$\begin{aligned} D = \mathbf{D} \cdot \mathbf{a}_D &= A_{\parallel}BC \sin(\theta_1 - \theta_2) \\ &= (B \sin\theta_1)(A_{\parallel}C \cos\theta_2) - (C \sin\theta_2)(A_{\parallel}B \cos\theta_1) \\ &= [\mathbf{B}(\mathbf{A}_{\parallel} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A}_{\parallel} \cdot \mathbf{B})] \cdot \mathbf{a}_D. \end{aligned}$$

The expression above does not alone guarantee the quantity inside the brackets to be $\mathbf{D}$, since the former may contain a vector that is normal to $\mathbf{D}$ (parallel to $\mathbf{A}_{\parallel}$); that is, $\mathbf{D} \cdot \mathbf{a}_D = \mathbf{E} \cdot \mathbf{a}_D$ does not guarantee $\mathbf{E} = \mathbf{D}$. In general, we can write

$$\mathbf{B}(\mathbf{A}_{\parallel} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A}_{\parallel} \cdot \mathbf{B}) = \mathbf{D} + k\mathbf{A}_{\parallel},$$

where k is a scalar quantity. To determine k, we scalar-multiply both sides of the above equation by $\mathbf{A}_{\parallel}$ and obtain

$$(\mathbf{A}_{\parallel} \cdot \mathbf{B})(\mathbf{A}_{\parallel} \cdot \mathbf{C}) - (\mathbf{A}_{\parallel} \cdot \mathbf{C})(\mathbf{A}_{\parallel} \cdot \mathbf{B}) = 0 = \mathbf{A}_{\parallel} \cdot \mathbf{D} + kA_{\parallel}^2.$$

Since $\mathbf{A}_{\parallel} \cdot \mathbf{D} = 0$, then $k = 0$ and

$$\mathbf{D} = \mathbf{B}(\mathbf{A}_{\parallel} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A}_{\parallel} \cdot \mathbf{B}),$$

which proves the back-cab rule, inasmuch as $\mathbf{A}_{\parallel} \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C}$ and $\mathbf{A}_{\parallel} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{B}$. ▬

Division by a vector is not defined, and expressions such as $k/\mathbf{A}$ and $\mathbf{B}/\mathbf{A}$ are meaningless.

2–4 Orthogonal Coordinate Systems

We have indicated before that although the laws of electromagnetism are invariant with coordinate system, solution of practical problems requires that the relations derived from these laws be expressed in a coordinate system appropriate to the geometry of the given problems. For example, if we are to determine the electric field at a certain point in space, we at least need to describe the position of the source and the location of this point in a coordinate system. In a three-dimensional space a point can be located as the intersection of three surfaces. Assume that the three families of surfaces are described by u_1 = constant, u_2 = constant, and u_3 = constant, where the u's need not all be lengths. (In the familiar Cartesian or rectangular coordinate system, u_1, u_2, and u_3 correspond to x, y, and z, respectively.) When these three surfaces are mutually perpendicular to one another, we have an ***orthogonal coordinate system.*** Nonorthogonal coordinate systems are not used because they complicate problems.

Some surfaces represented by u_i = constant (i = 1, 2, or 3) in a coordinate system may not be planes; they may be curved surfaces. Let $\mathbf{a}_{u_1}$, $\mathbf{a}_{u_2}$, and $\mathbf{a}_{u_3}$ be the unit vectors in the three coordinate directions. They are called the ***base vectors.*** In a general right-handed, orthogonal, curvilinear coordinate system the base vectors are arranged in such a way that the following relations are satisfied:

$$\mathbf{a}_{u_1} \times \mathbf{a}_{u_2} = \mathbf{a}_{u_3}, \tag{2–21a}$$

$$\mathbf{a}_{u_2} \times \mathbf{a}_{u_3} = \mathbf{a}_{u_1}, \tag{2–21b}$$

$$\mathbf{a}_{u_3} \times \mathbf{a}_{u_1} = \mathbf{a}_{u_2}. \tag{2–21c}$$

These three equations are not all independent, as the specification of one automatically implies the other two. We have, of course,

$$\mathbf{a}_{u_1} \cdot \mathbf{a}_{u_2} = \mathbf{a}_{u_2} \cdot \mathbf{a}_{u_3} = \mathbf{a}_{u_3} \cdot \mathbf{a}_{u_1} = 0 \tag{2–22}$$

and

$$\mathbf{a}_{u_1} \cdot \mathbf{a}_{u_1} = \mathbf{a}_{u_2} \cdot \mathbf{a}_{u_2} = \mathbf{a}_{u_3} \cdot \mathbf{a}_{u_3} = 1. \tag{2–23}$$

Any vector **A** can be written as the sum of its components in the three orthogonal directions, as follows:

$$\boxed{\mathbf{A} = \mathbf{a}_{u_1}A_{u_1} + \mathbf{a}_{u_2}A_{u_2} + \mathbf{a}_{u_3}A_{u_3}.} \tag{2–24}$$

From Eq. (2–24) the magnitude of **A** is

$$A = |\mathbf{A}| = (A_{u_1}^2 + A_{u_2}^2 + A_{u_3}^2)^{1/2}. \tag{2–25}$$

EXAMPLE 2–4 Given three vectors **A**, **B**, and **C**, obtain the expressions of (a) $\mathbf{A} \cdot \mathbf{B}$, (b) $\mathbf{A} \times \mathbf{B}$, and (c) $\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$ in the orthogonal curvilinear coordinate system (u_1, u_2, u_3).