CHAPTER 3

Electric Flux Density, Gauss's Law, and Divergence

After drawing a few of the fields described in the previous chapter and becoming familiar with the concept of the streamlines which show the direction of the force on a test charge at every point, it is difficult to avoid giving these lines a physical significance and thinking of them as *flux* lines. No physical particle is projected radially outward from the point charge, and there are no steel tentacles reaching out to attract or repel an unwary test charge, but as soon as the streamlines are drawn on paper there seems to be a picture showing "something" is present.

It is very helpful to invent an *electric flux* which streams away symmetrically from a point charge and is coincident with the streamlines and to visualize this flux wherever an electric field is present.

This chapter introduces and uses the concept of electric flux and electric flux density to solve again several of the problems presented in Chapter 2. The work here turns out to be much easier, and this is due to the extremely symmetrical problems which we are solving. ■

3.1 ELECTRIC FLUX DENSITY

About 1837 the Director of the Royal Society in London, Michael Faraday, became very interested in static electric fields and the effect of various insulating materials on these fields. This problem had been bothering him during the past ten years when he was experimenting in his now famous work on induced electromotive force, which we shall discuss in Chapter 10. With that subject completed, he had a pair of concentric metallic spheres constructed, the outer one consisting of two hemispheres that could be firmly clamped together. He also prepared shells of insulating material (or dielectric material, or simply *dielectric*) which would occupy the entire volume between the concentric spheres. We shall not make immediate use of his findings about dielectric materials, for we are restricting our attention to fields in free space until Chapter 6. At that time we shall see that the materials he used will be classified as ideal dielectrics.

His experiment, then, consisted essentially of the following steps:

1. With the equipment dismantled, the inner sphere was given a known positive charge.
2. The hemispheres were then clamped together around the charged sphere with about 2 cm of dielectric material between them.
3. The outer sphere was discharged by connecting it momentarily to ground.
4. The outer space was separated carefully, using tools made of insulating material in order not to disturb the induced charge on it, and the negative induced charge on each hemisphere was measured.

Faraday found that the total charge on the outer sphere was equal in *magnitude* to the original charge placed on the inner sphere and that this was true regardless of the dielectric material separating the two spheres. He concluded that there was some sort of "displacement" from the inner sphere to the outer which was independent of the medium, and we now refer to this flux as *displacement*, *displacement* flux, or simply *electric flux*.

Faraday's experiments also showed, of course, that a larger positive charge on the inner sphere induced a correspondingly larger negative charge on the outer sphere, leading to a direct proportionality between the electric flux and the charge on the inner sphere. The constant of proportionality is dependent on the system of units involved, and we are fortunate in our use of SI units, because the constant is unity. If electric flux is denoted by Ψ (psi) and the total charge on the inner sphere by Q, then for Faraday's experiment

$$\Psi = Q$$

and the electric flux Ψ is measured in coulombs.

We can obtain more quantitative information by considering an inner sphere of radius a and an outer sphere of radius b, with charges of Q and $-Q$, respectively (Figure 3.1). The paths of electric flux Ψ extending from the inner sphere to the outer sphere are indicated by the symmetrically distributed streamlines drawn radially from one sphere to the other.

At the surface of the inner sphere, Ψ coulombs of electric flux are produced by the charge $Q(=\Psi)$ coulombs distributed uniformly over a surface having an area of $4\pi a^2$ m^2. The density of the flux at this surface is $\Psi/4\pi a^2$ or $Q/4\pi a^2\,\text{C/m}^2$, and this is an important new quantity.

Electric flux density, measured in coulombs per square meter (sometimes described as "lines per square meter," for each line is due to one coulomb), is given the letter **D**, which was originally chosen because of the alternate names of *displacement flux density* or *displacement density*. Electric flux density is more descriptive, however, and we shall use the term consistently.

The electric flux density **D** is a vector field and is a member of the "flux density" class of vector fields, as opposed to the "force fields" class, which includes the electric

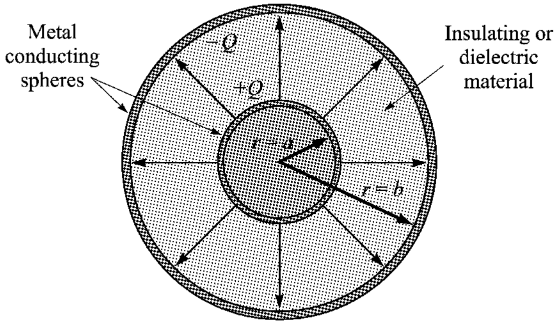

Figure 3.1 The electric flux in the region between a pair of charged concentric spheres. The direction and magnitude of D are not functions of the dielectric between the spheres.

field intensity **E**. The direction of **D** at a point is the direction of the flux lines at that point, and the magnitude is given by the number of flux lines crossing a surface normal to the lines divided by the surface area.

Referring again to Figure 3.1, the electric flux density is in the radial direction and has a value of

$$\mathbf{D}\Big|_{r=a} = \frac{Q}{4\pi a^2}\mathbf{a}_r \qquad \text{(inner sphere)}$$

$$\mathbf{D}\Big|_{r=b} = \frac{Q}{4\pi b^2}\mathbf{a}_r \qquad \text{(outer sphere)}$$

and at a radial distance r, where $a \le r \le b$,

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

If we now let the inner sphere become smaller and smaller, while still retaining a charge of Q, it becomes a point charge in the limit, but the electric flux density at a point r meters from the point charge is still given by

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r \tag{1}$$

for Q lines of flux are symmetrically directed outward from the point and pass through an imaginary spherical surface of area $4\pi r^2$.

This result should be compared with Section 2.2, Eq. (10), the radial electric field intensity of a point charge in free space,

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{a}_r$$

In free space, therefore,

$$\mathbf{D} = \epsilon_0 \mathbf{E} \quad \text{(free space only)} \tag{2}$$

Although (2) is applicable only to a vacuum, it is not restricted solely to the field of a point charge. For a general volume charge distribution in free space

$$\mathbf{E} = \int_{\text{vol}} \frac{\rho_\nu d\nu}{4\pi\epsilon_0 R^2} \mathbf{a}_R \quad \text{(free space only)} \tag{3}$$

where this relationship was developed from the field of a single point charge. In a similar manner, (1) leads to

$$\mathbf{D} = \int_{\text{vol}} \frac{\rho_\nu d\nu}{4\pi R^2} \mathbf{a}_R \tag{4}$$

and (2) is therefore true for any free-space charge configuration; we shall consider (2) as defining **D** in free space.

As a preparation for the study of dielectrics later, it might be well to point out now that, for a point charge embedded in an infinite ideal dielectric medium, Faraday's results show that (1) is still applicable, and thus so is (4). Equation (3) is not applicable, however, and so the relationship between **D** and **E** will be slightly more complicated than (2).

Since **D** is directly proportional to **E** in free space, it does not seem that it should really be necessary to introduce a new symbol. We do so for several reasons. First, **D** is associated with the flux concept, which is an important new idea. Second, the **D** fields we obtain will be a little simpler than the corresponding **E** fields, since ϵ_0 does not appear. And, finally, it helps to become a little familiar with **D** before it is applied to dielectric materials in Chapter 6.

Let us consider a simple numerical example to illustrate these new quantities and units.

EXAMPLE 3.1

We wish to find **D** in the region about a uniform line charge of 8 nC/m lying along the z axis in free space.

Solution. The **E** field is

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0 \rho}\mathbf{a}_\rho = \frac{8 \times 10^{-9}}{2\pi(8.854 \times 10^{-12})\rho}\mathbf{a}_\rho = \frac{143.8}{\rho}\mathbf{a}_\rho \text{ V/m}$$

At $\rho = 3\text{m}$, $\mathbf{E} = 47.9\mathbf{a}_\rho$ V/m.

Associated with the **E** field, we find

$$\mathbf{D} = \frac{\rho_L}{2\pi\rho}\mathbf{a}_\rho = \frac{8 \times 10^{-9}}{2\pi\rho}\mathbf{a}_\rho = \frac{1.273 \times 10^{-9}}{\rho}\mathbf{a}_\rho \text{ C/m}^2$$

The value at $\rho = 3$ m is $\mathbf{D} = 0.424\mathbf{a}_\rho$ nC/m.

The total flux leaving a 5-m length of the line charge is equal to the total charge on that length, or $\Psi = 40$ nC.

D3.1. Given a 60-μC point charge located at the origin, find the total electric flux passing through: (*a*) that portion of the sphere $r = 26$ cm bounded by $0 < \theta < \frac{\pi}{2}$ and $0 < \phi < \frac{\pi}{2}$; (*b*) the closed surface defined by $\rho = 26$ cm and $z = \pm 26$ cm; (*c*) the plane $z = 26$ cm.

Ans. 7.5 μC; 60 μC; 30 μC

D3.2. Calculate **D** in rectangular coordinates at point $P(2, -3, 6)$ produced by: (*a*) a point charge $Q_A = 55$ mC at $Q(-2, 3, -6)$; (*b*) a uniform line charge $\rho_{LB} = 20$ mC/m on the x axis; (*c*) a uniform surface charge density $\rho_{SC} = 120\ \mu\text{C/m}^2$ on the plane $z = -5$ m.

Ans. $6.38\mathbf{a}_x - 9.57\mathbf{a}_y + 19.14\mathbf{a}_z\ \mu\text{C/m}^2$; $-212\mathbf{a}_y + 424\mathbf{a}_z\ \mu\text{C/m}^2$; $60\mathbf{a}_z\ \mu\text{C/m}^2$

3.2 GAUSS'S LAW

The results of Faraday's experiments with the concentric spheres could be summed up as an experimental law by stating that the electric flux passing through any imaginary spherical surface lying between the two conducting spheres is equal to the charge enclosed within that imaginary surface. This enclosed charge is distributed on the surface of the inner sphere, or it might be concentrated as a point charge at the center of the imaginary sphere. However, since one coulomb of electric flux is produced by one coulomb of charge, the inner conductor might just as well have been a cube or a brass door key and the total induced charge on the outer sphere would still be the same. Certainly the flux density would change from its previous symmetrical distribution to some unknown configuration, but $+Q$ coulombs on any inner conductor would produce an induced charge of $-Q$ coulombs on the surrounding sphere. Going one step further, we could now replace the two outer hemispheres by an empty (but completely closed) soup can. Q coulombs on the brass door key would produce $\Psi = Q$ lines of electric flux and would induce $-Q$ coulombs on the tin can[1]

These generalizations of Faraday's experiment lead to the following statement, which is known as *Gauss's law*:

> The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

The contribution of Gauss, one of the greatest mathematicians the world has ever produced, was actually not in stating the law as we have, but in providing a mathematical form for this statement, which we shall now obtain.

[1] If it were a perfect insulator, the soup could even be left in the can without any difference in the results.

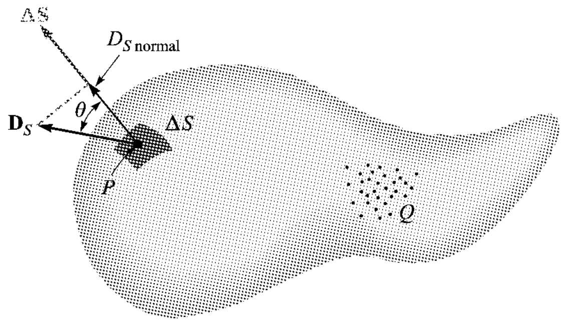

Figure 3.2 The electric flux density D_S at P due to charge Q. The total flux passing through ΔS is $D_S \cdot \Delta S$.

Let us imagine a distribution of charge, shown as a cloud of point charges in Figure 3.2, surrounded by a closed surface of any shape. The closed surface may be the surface of some real material, but more generally it is any closed surface we wish to visualize. If the total charge is Q, then Q coulombs of electric flux will pass through the enclosing surface. At every point on the surface the electric-flux-density vector $\mathbf{D}$ will have some value $\mathbf{D}_S$, where the subscript S merely reminds us that $\mathbf{D}$ must be evaluated at the surface, and $\mathbf{D}_S$ will in general vary in magnitude and direction from one point on the surface to another.

We must now consider the nature of an incremental element of the surface. An incremental element of area ΔS is very nearly a portion of a plane surface, and the complete description of this surface element requires not only a statement of its magnitude ΔS but also of its orientation in space. In other words, the incremental surface element is a vector quantity. The only unique direction which may be associated with $\Delta\mathbf{S}$ is the direction of the normal to that plane which is tangent to the surface at the point in question. There are, of course, two such normals, and the ambiguity is removed by specifying the outward normal whenever the surface is closed and "outward" has a specific meaning.

At any point P consider an incremental element of surface ΔS and let $\mathbf{D}_S$ make an angle θ with $\Delta\mathbf{S}$, as shown in Figure 3.2. The flux crossing ΔS is then the product of the normal component of $\mathbf{D}_S$ and $\Delta\mathbf{S}$,

$$\Delta\Psi = \text{flux crossing } \Delta S = D_{S,norm}\Delta S = D_S \cos\theta \,\Delta S = \mathbf{D}_S \cdot \Delta\mathbf{S}$$

where we are able to apply the definition of the dot product developed in Chapter 1.

The *total* flux passing through the closed surface is obtained by adding the differential contributions crossing each surface element $\Delta\mathbf{S}$,

$$\Psi = \int d\Psi = \oint_{\substack{\text{closed}\\ \text{surface}}} \mathbf{D}_S \cdot d\mathbf{S}$$

The resultant integral is a *closed surface integral*, and since the surface element $d\mathbf{S}$ always involves the differentials of two coordinates, such as $dx\,dy$, $\rho\,d\phi\,d\rho$, or $r^2 \sin\theta \, d\theta \, d\phi$, the integral is a double integral. Usually only one integral sign is used for brevity, and we shall always place an S below the integral sign to indicate a surface integral, although this is not actually necessary since the differential $d\mathbf{S}$ is automatically the signal for a surface integral. One last convention is to place a small circle on the integral sign itself to indicate that the integration is to be performed over a *closed* surface. Such a surface is often called a *gaussian surface*. We then have the mathematical formulation of Gauss's law,

$$\Psi = \oint_S \mathbf{D}_S \cdot d\mathbf{S} = \text{charge enclosed} = Q \tag{5}$$

The charge enclosed might be several point charges, in which case

$$Q = \Sigma Qn$$

or a line charge,

$$Q = \int \rho_L \, dL$$

or a surface charge,

$$Q = \int_S \rho_S dS \qquad \text{(not necessarily a closed surface)}$$

or a volume charge distribution,

$$Q = \int_{\text{vol}} \rho_v \, dv$$

The last form is usually used, and we should agree now that it represents any or all of the other forms. With this understanding, Gauss's law may be written in terms of the charge distribution as

$$\oint_S \mathbf{D}_S \cdot d\mathbf{S} = \int_{\text{vol}} \rho_v \, dv \tag{6}$$

a mathematical statement meaning simply that the total electric flux through any closed surface is equal to the charge enclosed.

To illustrate the application of Gauss's law, let us check the results of Faraday's experiment by placing a point charge Q at the origin of a spherical coordinate system (Figure 3.3) and by choosing our closed surface as a sphere of radius a. The electric field intensity of the point charge has been found to be

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{a}_r$$

and since

$$\mathbf{D} = \epsilon_0 \mathbf{E}$$

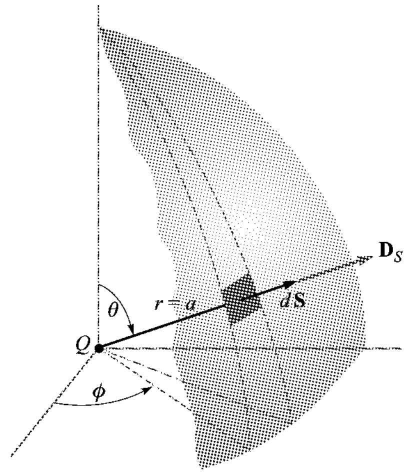

Figure 3.3 Application of Gauss's law to the field of a point charge Q on a spherical closed surface of radius a. The electric flux density D is everywhere normal to the spherical surface and has a constant magnitude at every point on it.

we have, as before,

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

At the surface of the sphere,

$$\mathbf{D}_S = \frac{Q}{4\pi a^2}\mathbf{a}_r$$

The differential element of area on a spherical surface is, in spherical coordinates from Chapter 1,

$$dS = r^2 \sin\theta \, d\theta \, d\phi = a^2 \sin\theta \, d\theta \, d\phi$$

or

$$d\mathbf{S} = a^2 \sin\theta \, d\theta \, d\phi \, \mathbf{a}_r$$

The integrand is

$$\mathbf{D}_S \cdot d\mathbf{S} = \frac{Q}{4\pi a^2} a^2 \sin\theta \, d\theta \, d\phi \mathbf{a}_r \cdot \mathbf{a}_r = \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

leading to the closed surface integral

$$\int_{\phi=0}^{\phi=2\pi} \int_{\theta=\phi}^{\theta=\pi} \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

where the limits on the integrals have been chosen so that the integration is carried over the entire surface of the sphere once.[2] Integrating gives

$$\int_0^{2\pi} \frac{Q}{4\pi}\left(-\cos\theta\right)_0^{\pi} d\phi = \int_0^{2\pi} \frac{Q}{2\pi} d\phi = Q$$

Animations

and we obtain a result showing that Q coulombs of electric flux are crossing the surface, as we should since the enclosed charge is Q coulombs.

The following section contains examples of the application of Gauss's law to problems of a simple symmetrical geometry with the object of finding the electric field intensity.

D3.3. Given the electric flux density, $\mathbf{D} = 0.3r^2\mathbf{a}_r$ nC/m^2 in free space: (*a*) find $\mathbf{E}$ at point $P(r = 2, \theta = 25^\circ, \phi = 90^\circ)$; (*b*) find the total charge within the sphere $r = 3$; (*c*) find the total electric flux leaving the sphere $r = 4$.

Ans. $135.5\mathbf{a}_r$ V/m; 305 nC; 965 nC

D3.4. Calculate the total electric flux leaving the cubical surface formed by the six planes $x, y, z = \pm 5$ if the charge distribution is: (*a*) two point charges, 0.1 μC at $(1, -2, 3)$ and $\frac{1}{7}$ μC at $(-1, 2, -2)$; (*b*) a uniform line charge of π μC/m at $x = -2, y = 3$; (*c*) a uniform surface charge of 0.1 μC/m^2 on the plane $y = 3x$.

Ans. 0.243 μC; 31.4 μC; 10.54 μC

3.3 APPLICATION OF GAUSS'S LAW: SOME SYMMETRICAL CHARGE DISTRIBUTIONS

Let us now consider how we may use Gauss's law,

$$Q = \oint_S \mathbf{D}_S \cdot d\mathbf{S}$$

to determine $\mathbf{D}_S$ if the charge distribution is known. This is an example of an integral equation in which the unknown quantity to be determined appears inside the integral.

The solution is easy if we are able to choose a closed surface which satisfies two conditions:

1. $\mathbf{D}_S$ is everywhere either normal or tangential to the closed surface, so that $\mathbf{D}_S \cdot d\mathbf{S}$ becomes either $D_S dS$ or zero, respectively.
2. On that portion of the closed surface for which $\mathbf{D}_S \cdot d\mathbf{S}$ is not zero, $D_S =$ constant.

[2] Note that if θ and ϕ both cover the range from 0 to 2π, the spherical surface is covered twice.

This allows us to replace the dot product with the product of the scalars D_S and dS and then to bring D_S outside the integral sign. The remaining integral is then $\int_S dS$ over that portion of the closed surface which $\mathbf{D}_S$ crosses normally, and this is simply the area of this section of that surface.

Only a knowledge of the symmetry of the problem enables us to choose such a closed surface, and this knowledge is obtained easily by remembering that the electric field intensity due to a positive point charge is directed radially outward from the point charge.

Let us again consider a point charge Q at the origin of a spherical coordinate system and decide on a suitable closed surface which will meet the two requirements previously listed. The surface in question is obviously a spherical surface, centered at the origin and of any radius r. $\mathbf{D}_S$ is everywhere normal to the surface; D_S has the same value at all points on the surface.

Then we have, in order,

$$\begin{aligned} Q &= \oint_S \mathbf{D}_S \cdot d\mathbf{S} = \oint_{\text{sph}} D_S dS \\ &= D_S \oint_{\text{sph}} dS = D_S \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} r^2 \sin\theta \, d\theta \, d\phi \\ &= 4\pi r^2 D_S \end{aligned}$$

and hence

$$D_S = \frac{Q}{4\pi r^2}$$

Since r may have any value and since $\mathbf{D}_S$ is directed radially outward,

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r \qquad \mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{a}_r$$

which agrees with the results of Chapter 2. The example is a trivial one, and the objection could be raised that we had to know that the field was symmetrical and directed radially outward before we could obtain an answer. This is true, and that leaves the inverse-square-law relationship as the only check obtained from Gauss's law. The example does, however, serve to illustrate a method which we may apply to other problems, including several to which Coulomb's law is almost incapable of supplying an answer.

Are there any other surfaces which would have satisfied our two conditions? The student should determine that such simple surfaces as a cube or a cylinder do not meet the requirements.

As a second example, let us reconsider the uniform line charge distribution ρ_L lying along the z axis and extending from $-\infty$ to $+\infty$. We must first obtain a knowledge