Chapter 3

Discrete-Time Signals in the Frequency Domain

In Section 2.4.3, we pointed out that any arbitrary sequence can be represented in the time domain as a weighted linear combination of delayed unit sample sequences $\{\delta[n-k]\}$. An important consequence of this representation, derived in Section 4.4.1, is the input–output characterization of a certain class of discrete-time systems in the time domain. In many applications, it is convenient to consider an alternate description of a sequence in terms of complex exponential sequences of the form $\{e^{j\omega n}\}$, where ω is the normalized frequency variable in radians. This leads to a particularly useful representation of discrete-time sequences and certain discrete-time systems in the frequency domain.[1]

The frequency-domain representation of a discrete-time sequence discussed in this chapter is the discrete-time Fourier transform by which a time-domain sequence is mapped into a continuous function of the frequency variable ω. Because of the periodicity of the discrete-time Fourier transform, the corresponding discrete-time sequence can be simply obtained by computing its Fourier series representation. Since the representation is in terms of an infinite series, existence of the discrete-time Fourier transform is examined along with its properties.

We next derive the conditions under which a continuous-time signal can be represented uniquely by a discrete-time signal, and also show how the original continuous-time signal can be recovered from its discrete-time equivalent. The conversion of a continuous-time signal into an equivalent discrete-time signal is performed by a sample-and-hold circuit followed by an analog-to-digital converter. The effect of a practical sample-and-hold circuit on the generation of the discrete-time equivalent is discussed.

In this chapter also, we make extensive use of MATLAB to illustrate through computer simulations the various concepts introduced.

3.1 The Continuous-Time Fourier Transform

We begin with a brief review of the continuous-time Fourier transform, a frequency-domain representation of a continuous-time signal, and its properties, as it will provide a better understanding of the frequency-domain representation of the discrete-time signals and systems, in addition to pointing out the major differences between these two transforms.

[1] Periodic sequences can be represented in the frequency domain by means of the discrete Fourier series (see Problem 5.3).

3.1.1 Definition

The frequency-domain representation of a continuous-time signal $x_a(t)$ is given by the *continuous-time Fourier transform* (CTFT):

$$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt. \tag{3.1}$$

The CTFT often is referred to as the *Fourier spectrum*, or simply, the *spectrum* of the continuous-time signal. The continuous-time signal $x_a(t)$ can be recovered from its CTFT $X_a(j\Omega)$ via the inverse *Fourier integral*

$$x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(j\Omega) e^{j\Omega t} d\Omega. \tag{3.2}$$

We denote the CTFT pair of Eqs. (3.1) and (3.2) as

$$x_a(t) \overset{\text{CTFT}}{\longleftrightarrow} X_a(j\Omega).$$

In Eqs. (3.1) and (3.2), Ω is real and denotes the continuous-time angular frequency variable in radians per sec. The inverse Fourier transform given by Eq. (3.2) can be interpreted as a linear combination of infinitesimally small complex exponential signals of the form $\frac{1}{2\pi} e^{j\Omega t} d\Omega$, weighted by the complex constant $X_a(j\Omega)$ over the angular frequency range from $-\infty$ to ∞. It can be seen from the definition given by Eq. (3.1) that, in general, the CTFT is a complex function of Ω in the range $-\infty < \Omega < \infty$. It can be expressed in polar form as

$$X_a(j\Omega) = |X_a(j\Omega)| e^{j\theta_a(\Omega)},$$

where

$$\theta_a(\Omega) = \arg\{X_a(j\Omega)\}.$$

The quantity $|X_a(j\Omega)|$ is called the *magnitude spectrum*, and the quantity $\theta_a(\Omega)$ is called the *phase spectrum*, with both spectra being real functions of Ω.

In general, the CTFT $X_a(j\Omega)$ defined by Eq. (3.1) exists if the continuous-time signal $x_a(t)$ satisfies the *Dirichlet conditions:*

(a) The signal has a finite number of finite discontinuities and a finite number of maxima and minima in any finite interval.

(b) The signal is absolutely integrable; that is,

$$\int_{-\infty}^{\infty} |x_a(t)| dt < \infty. \tag{3.3}$$

If the Dirichlet conditions are satisfied, the integral on the right-hand side of Eq. (3.2) converges to $x_a(t)$ at all values of t except at values of t where $x_a(t)$ has discontinuities.

It easily can be shown that if $x_a(t)$ is absolutely integrable, then $|X_a(j\Omega)| < \infty$, proving the existence of the CTFT (Problem 3.1). We illustrate the CTFT computation in Examples 3.1- 3.3.

EXAMPLE 3.1 An Absolutely Integrable Continuous-Time Signal

Consider the real signal

$$x_a(t) = \begin{cases} e^{-\alpha t}, & t \geq 0, \\ 0, & t < 0, \end{cases} \tag{3.4}$$

where $0 < \alpha < \infty$. A plot of the above signal for $\alpha = 0.5$/sec is shown in Figure 3.1. This function is absolutely integrable as

$$\int_{-\infty}^{\infty} |x_a(t)| dt = \int_{0}^{\infty} e^{-\alpha t} dt = -\frac{e^{-\alpha t}}{\alpha}\Big|_0^{\infty} = \frac{1}{\alpha} < \infty.$$

Its CTFT obtained using Eq. (3.1) is given by

$$\begin{aligned} X_a(j\Omega) &= \int_{0}^{\infty} e^{-\alpha t} e^{-j\Omega t} dt = \int_{0}^{\infty} e^{-(\alpha + j\Omega)t} dt \\ &= -\frac{1}{\alpha + j\Omega} e^{-(\alpha+j\Omega)t}\Big|_0^{\infty} = \frac{1}{\alpha + j\Omega}. \end{aligned} \tag{3.5}$$

We can express the above CTFT as

$$X_a(j\Omega) = \frac{1}{\sqrt{\alpha^2 + \Omega^2}} e^{-j\tan^{-1}(\Omega/\alpha)},$$

where $|X_a(j\Omega)| = 1/(\sqrt{\alpha^2 + \Omega^2})$ is the magnitude spectrum and $\theta_a(\Omega) = -\tan^{-1}(\Omega/\alpha)$ is the phase spectrum. A plot of these two functions is shown in Figure 3.2.

It can be shown that for $\alpha < 0$, $x_a(t)$ is not absolutely integrable, and hence, the CTFT $X_a(j\Omega)$ does not exist in this case.

EXAMPLE 3.2 Continuous-Time Fourier Transform of an Impulse Function

We determine the CTFT $\Delta(j\Omega)$ of an ideal impulse $\delta(t)$. Applying Eq. (3.1), we get

$$\Delta(j\Omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\Omega t} dt = 1,$$

using the sampling property of the delta function.

EXAMPLE 3.3 Continuous-Time Fourier Transform of a Shifted Impulse Function

Consider the shifted impulse function $x_a(t) = \delta(t - t_o)$. Its CTFT is thus given by

$$X_a(j\Omega) = \int_{-\infty}^{\infty} \delta(t - t_o) e^{-j\Omega t} dt = e^{-j\Omega t_o}.$$

It should be noted that an absolutely integrable continuous-time signal $x_a(t)$ with bounded amplitude always has finite energy; that is, Eq. (3.6) holds:

$$\int_{-\infty}^{\infty} |x_a(t)|^2 dt < \infty. \tag{3.6}$$

However, the CTFT may exist for a finite-energy continuous-time signal that is not absolutely integrable (Problem 3.5). Hence, Eq. (3.6) is a milder condition than that given by Eq. (3.3).

The CTFT can also be defined using ideal impulses for some functions that do not satisfy either Eq. (3.3) or Eq. (3.6).

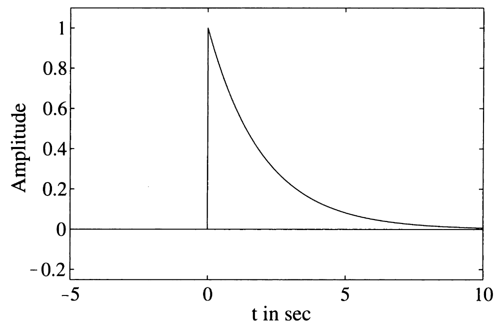

Figure 3.1: Plot of the continuous-time function of Eq. (3.4) for $\alpha = 0.5/\text{sec}$.

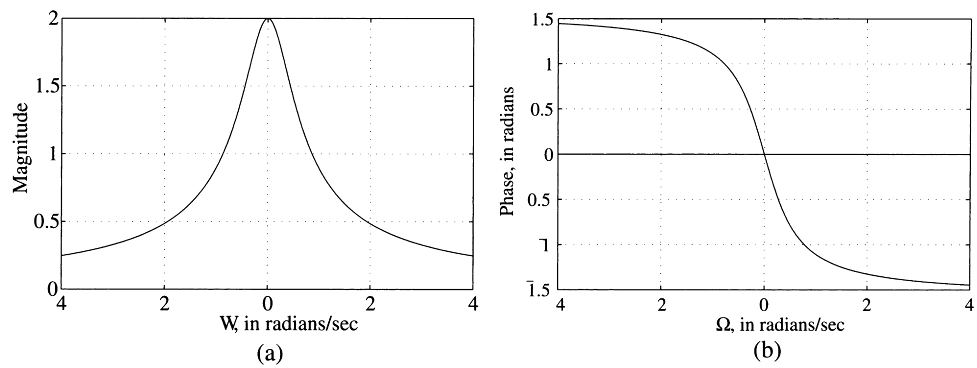

Figure 3.2: (a) Magnitude and (b) phase of $X_a(j\Omega) = 1/(0.5/\text{sec} + j\Omega)$.

3.1.2 Energy Density Spectrum

The total energy $\mathcal{E}_x$ of a finite-energy continuous-time complex signal $x_a(t)$ is given by

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x_a(t)|^2 \, dt = \int_{-\infty}^{\infty} x_a(t) x_a^*(t) \, dt. \tag{3.7}$$

The energy can also be expressed in terms of the CTFT $X_a(j\Omega)$. To this end, we first replace $x_a^*(t)$ in the above equation by its inverse CTFT expression as shown below inside the square brackets:

$$\mathcal{E}_x = \int_{-\infty}^{\infty} x_a(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) e^{-j\Omega t} \, d\Omega \right] dt.$$

Interchanging the order of the integrations in the above, we get

$$\begin{aligned} \mathcal{E}_x &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) \left[\int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} \, dt \right] d\Omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) X_a(j\Omega) \, d\Omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 \, d\Omega. \end{aligned} \tag{3.8}$$

Combining Eqs. (3.7) and (3.8), we arrive at

$$\int_{-\infty}^{\infty} |x_a(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_a(j\Omega)|^2\, d\Omega, \tag{3.9}$$

which is more commonly known as the *Parseval's theorem* for finite-energy continuous-time signals. Example 3.4 illustrates the application of the above equation.

EXAMPLE 3.4 Energy of a Continuous-Time Signal

We determine the total energy $\mathcal{E}_x$ of the continuous-time signal of Eq. (3.4). Using Eq. (3.9), we have

$$\mathcal{E}_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\alpha^2 + \Omega^2}\, d\Omega = \frac{1}{2\pi}\cdot\frac{\pi}{\alpha} = \frac{1}{2\alpha}.$$

For $\alpha = 0.5$, the total energy is $\mathcal{E}_x = 1$.

The integrand $|X_a(j\Omega)|^2$ on the right-hand side of Eq. (3.8) is called the *energy density spectrum* of the continuous-time signal $x_a(t)$ and is usually denoted by the symbol $S_{xx}(\Omega)$; that is,

$$S_{xx}(\Omega) = |X_a(j\Omega)|^2.$$

The energy $\mathcal{E}_{x,r}$ over a specified range of frequencies $\Omega_a \le \Omega \le \Omega_b$ of the signal $x_a(t)$ can be computed by integrating $S_{xx}(\Omega)$ over this range:

$$\mathcal{E}_{x,r} = \frac{1}{2\pi}\int_{\Omega_a}^{\Omega_b} S_{xx}(\Omega)\, d\Omega.$$

3.1.3 Band-Limited Continuous-Time Signals

A full-band, finite-energy, continuous-time signal has a spectrum occupying the whole frequency range $-\infty < \Omega < \infty$, whereas a *band-limited* continuous-time signal has a spectrum that is limited to a portion of the above frequency range. An ideal band-limited signal has a spectrum that is zero outside a finite frequency range $\Omega_a \le |\Omega| \le \Omega_b$; that is,

$$X_a(j\Omega) = \begin{cases} 0, & 0 \le |\Omega| < \Omega_a, \\ 0, & \Omega_b < |\Omega| < \infty. \end{cases}$$

However, an ideal band-limited signal cannot be generated in practice, and, for practical purposes, it is sufficient to ensure that for a band-limited signal the signal energy is sufficiently small outside the specified frequency range.

Band-limited signals are classified according to the frequency range where most of the signal's energy is concentrated. A *lowpass* continuous-time signal has a spectrum occupying the frequency range $0 \le |\Omega| \le \Omega_p < \infty$, where Ω_p is called the *bandwidth* of the signal. Likewise, a *highpass* continuous-time signal has a spectrum occupying the frequency range $0 < \Omega_p \le |\Omega| < \infty$, where the *bandwidth* of the signal is from Ω_p to ∞. Finally, a *bandpass* continuous-time signal has a spectrum occupying the frequency range $0 < \Omega_L \le |\Omega| \le \Omega_H < \infty$, where $\Omega_H - \Omega_L$ is its *bandwidth*. A precise definition of the bandwidth depends on applications. As can be seen from Figure 3.2(a), the continuous-time signal of Eq. (3.4) is a lowpass signal. It can be shown that 80% of the energy of this signal is contained in the frequency range $0 \le |\Omega| \le 0.4898\pi$, and hence, we can define the 80% bandwidth of the signal to be 0.4898π radians (Problem 3.10).

3.2 The Discrete-Time Fourier Transform

The frequency-domain representation of a discrete-time sequence is given by the *discrete-time Fourier transform* (DTFT), which expresses the sequence as a weighted combination of the complex exponential sequence $\{e^{j\omega n}\}$ where ω is the real normalized frequency variable. If there is no ambiguity, for brevity often the discrete-time Fourier transform is called simply the *Fourier transform* (FT). The Fourier transform representation of a sequence, if it exists, is unique, and the original sequence can be computed from its transform representation by an inverse transform operation. We first define the forward transform and derive its inverse transform. We then describe the condition for its existence and review its important properties.

3.2.1 Definition

The *discrete-time Fourier transform* $X(e^{j\omega})$ of a sequence $x[n]$ is defined by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}. \tag{3.10}$$

We illustrate the Fourier transform computation in Examples 3.5 and 3.6.

EXAMPLE 3.5 Discrete-Time Fourier Transform of the Unit Sample Sequence

Consider the unit sample sequence $\delta[n]$ defined in Eq. (2.45). Its Fourier transform $\Delta(e^{j\omega})$ obtained using Eq. (3.10) is given by

$$\Delta(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]\, e^{-j\omega n} = 1, \tag{3.11}$$

where we have used the sampling property of the unit sample sequence; that is, $\delta[0] = 1$, and $\delta[n] = 0$ for $n \neq 0$.

EXAMPLE 3.6 Discrete-Time Fourier Transform of an Exponential Sequence

Consider the causal exponential sequence

$$x[n] = \alpha^n \mu[n], \quad |\alpha| < 1, \tag{3.12}$$

shown in Figure 3.3 for $\alpha = 0.5$. Its Fourier transform $X(e^{j\omega})$ obtained using Eq. (3.10) is given by

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} (\alpha\, e^{-j\omega})^n = \frac{1}{1-\alpha\, e^{-j\omega}}, \end{aligned} \tag{3.13}$$

as $|\alpha e^{-j\omega}| = |\alpha| < 1$.

It should be noted here that the Fourier transforms of most practical discrete-time sequences can be expressed in terms of a sum of a convergent geometric series, which can be summed in a simple closed form as illustrated by Example 3.6. We take up the issue of the convergence of a general Fourier transform in Section 3.2.4.

As can be seen from the definition, the discrete-time Fourier transform $X(e^{j\omega})$ of a sequence $x[n]$ is a function of the normalized angular frequency ω. However, unlike the continuous-time Fourier transform,

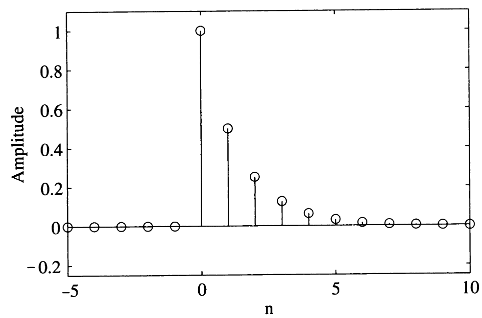

Figure 3.3: Plot of the discrete-time sequence of Eq. (3.12) for $\alpha = 0.5$.

it is a periodic function in ω with a period 2π. To verify this latter property, observe that for any integer k,

$$\begin{aligned} X(e^{j(\omega+2\pi k)}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2\pi k)n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}e^{-j2\pi kn} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X(e^{j\omega}), \qquad \text{for all values of } k, \end{aligned}$$

where we have used the fact $e^{-j2\pi kn} = 1$. It therefore follows that Eq. (3.10) represents the Fourier series expansion of the periodic function $X(e^{j\omega})$. As a result, the Fourier coefficients $x[n]$ can be computed from $X(e^{j\omega})$ using the Fourier integral given by

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\, d\omega, \tag{3.14}$$

called the *inverse discrete-time Fourier transform*. It should be noted that even though the integration in Eq. (3.14) can be carried out over any interval of duration 2π, it is a common practice to choose the interval $[-\pi, \pi]$. The inverse discrete-time Fourier transform given by Eq. (3.14) can be interpreted as a linear combination of infinitesimally small complex exponential signals of the form $\frac{1}{2\pi}e^{j\omega n}d\omega$, weighted by the complex constant $X(e^{j\omega})$ over the normalized angular frequency range from $-\pi$ to π.

Equations (3.10) and (3.14) constitute a discrete-time Fourier transform pair for the sequence $x[n]$. Equation (3.10) usually is referred to as the *analysis equation*, because it analyzes how much of each complex exponential signal is present in the original signal. On the other hand, Eq. (3.14) is referred to as the *synthesis equation*, because it synthesizes an arbitrary signal from its complex exponential components. For notational convenience, we shall use the operator symbol

$$\mathcal{F}\{x[n]\}$$

to denote the $X(e^{j\omega})$ of the sequence $x[n]$. Likewise, we shall use the operator symbol

$$\mathcal{F}^{-1}\{X(e^{j\omega})\}$$

to denote the inverse Fourier transform $x[n]$ of the transform $X(e^{j\omega})$. A discrete-time Fourier transform pair will be denoted as

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}). \tag{3.15}$$

To verify that the integral on the right-hand side of Eq. (3.14) indeed results in the inverse FT $x[n]$, we substitute the expression for $X(e^{j\omega})$ from Eq. (3.10) in Eq. (3.14), arriving at

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{\ell=-\infty}^{\infty} x[\ell]\,e^{-j\omega\ell}\right) e^{j\omega n}\,d\omega.$$

The order of integration and the summation on the right-hand side of the above equation can be interchanged if the summation inside the brackets converges uniformly; that is, if $X(e^{j\omega})$ exists. Under this condition, we get from the above

$$\begin{aligned}\sum_{\ell=-\infty}^{\infty} x[\ell]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-\ell)}\,d\omega\right) &= \sum_{\ell=-\infty}^{\infty} x[\ell]\,\frac{\sin\pi(n-\ell)}{\pi(n-\ell)}\\ &= \sum_{\ell=-\infty}^{\infty} x[\ell]\,\mathrm{sinc}(n-\ell).\end{aligned}$$

For $n \neq \ell$, $\sin\pi(n-\ell) = 0$, and as a result, $\mathrm{sinc}(n-\ell) = 0$. For $n = \ell$, $\mathrm{sinc}(n-\ell) = 0/0$, which is undefined. To determine the correct value we observe that $\mathrm{sinc}(n) = \sin(\pi n)/\pi n$ is the sampled value of the continuous-time function $\sin(\pi t)/\pi t$ for $t = n$; that is, $\sin(\pi n)/\pi n = \frac{\sin(\pi t)}{\pi t}\big|_{t=n}$. Using l'Hôpital's rule we get

$$\lim_{t\to 0}\frac{\sin(\pi t)}{\pi t} = \lim_{t\to 0}\frac{\pi\cos(\pi t)}{\pi} = 1.$$

Therefore

$$\begin{aligned}\mathrm{sinc}(n-\ell) &= \begin{cases} 1, & n = \ell,\\ 0, & n \neq \ell,\end{cases}\\ &= \delta[n-\ell].\end{aligned} \tag{3.16}$$

Hence,

$$\sum_{\ell=-\infty}^{\infty} x[\ell]\,\mathrm{sinc}(n-\ell) = \sum_{\ell=-\infty}^{\infty} x[\ell]\,\delta[n-\ell] = x[n],$$

using the sampling property of the unit sample sequence.

3.2.2 Basic Properties

We have already demonstrated in Section 3.2.1 one basic property of the Fourier transform, namely, the periodicity property of the transform. Here we examine a few additional basic properties of the Fourier transform of a complex sequence.

In general, the Fourier transform $X(e^{j\omega})$ is a complex function of the real variable ω and can be written in rectangular form as

$$X(e^{j\omega}) = X_{\mathrm{re}}(e^{j\omega}) + jX_{\mathrm{im}}(e^{j\omega}), \tag{3.17}$$

where $X_{\mathrm{re}}(e^{j\omega})$ and $X_{\mathrm{im}}(e^{j\omega})$ are, respectively, the real and imaginary parts of $X(e^{j\omega})$, and are real functions of ω. From Eq. (3.17), it follows that

$$X_{\mathrm{re}}(e^{j\omega}) = \tfrac{1}{2}\{X(e^{j\omega}) + X^*(e^{j\omega})\}, \tag{3.18a}$$

$$X_{\mathrm{im}}(e^{j\omega}) = \tfrac{1}{2j}\{X(e^{j\omega}) - X^*(e^{j\omega})\}, \tag{3.18b}$$

where $X^*(e^{j\omega})$ denotes the complex conjugate of $X(e^{j\omega})$.

The Fourier transform $X(e^{j\omega})$ can alternately be expressed in the polar form as

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\theta(\omega)}, \tag{3.19}$$

where

$$\theta(\omega) = \arg\{X(e^{j\omega})\}. \tag{3.20}$$

The quantity $|X(e^{j\omega})|$ is called the *magnitude function*, and the quantity $\theta(\omega)$ is called the *phase function*, with both functions again being real functions of ω. In many applications, the Fourier transform is called the *Fourier spectrum*, and likewise, $|X(e^{j\omega})|$ and $\theta(\omega)$ are referred to as the *magnitude spectrum* and *phase spectrum*, respectively.

The relations between the rectangular and polar forms of $X(e^{j\omega})$ follow from Eqs. (3.17) and (3.19) and are given by

$$X_{\text{re}}(e^{j\omega}) = |X(e^{j\omega})| \cos\, \theta(\omega), \tag{3.21a}$$

$$X_{\text{im}}(e^{j\omega}) = |X(e^{j\omega})| \sin\, \theta(\omega), \tag{3.21b}$$

$$|X(e^{j\omega})|^2 = X(e^{j\omega})X^*(e^{j\omega}) = X_{\text{re}}^2(e^{j\omega}) + X_{\text{im}}^2(e^{j\omega}), \tag{3.21c}$$

$$\tan\theta(\omega) = \frac{X_{\text{im}}(e^{j\omega})}{X_{\text{re}}(e^{j\omega})}. \tag{3.21d}$$

As in the case of the continuous-time Fourier transform, the phase function is also not uniquely specified for the discrete-time Fourier transform. Note from Eq. (3.19) that if we replace $\theta(\omega)$ with $\theta(\omega)+2\pi k$, where k is any integer, we get

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j[\theta(\omega)+2\pi k]} = |X(e^{j\omega})|e^{j\theta(\omega)},$$

indicating that the Fourier transform $X(e^{j\omega})$ remains unchanged. Consequently, the phase function $\theta(\omega)$ cannot be uniquely specified for any discrete-time Fourier transform for all values of ω. Unless otherwise stated, we will assume that the phase function $\theta(\omega)$ is restricted to the following range of values,

$$-\pi \le \theta(\omega) < \pi,$$

called the *principal value*.

3.2.3 Symmetry Relations

We review here some additional properties of the Fourier transform that are based on the symmetry relations. These properties can simplify the computational complexity and are often useful in digital signal processing applications. We list in Table 3.1 the symmetry relations of the Fourier transform of a real sequence and in Table 3.2 the symmetry relations of the Fourier transform of a complex sequence. The proofs of these symmetry relations are quite straightforward and have been left as exercises (Problems 3.36 and 3.37).

An interesting application of the symmetry property of a real signal is in the computation of its magnitude function. Consider the DTFT $X(e^{j\omega})$. Now, from Eq. (3.21c) $|X(e^{j\omega})|^2 = X(e^{j\omega})X^*(e^{j\omega})$. For a real signal it can be seen from Table 3.1 that $X^*(e^{j\omega}) = X(e^{-j\omega})$. Hence, the magnitude function of a real signal can be easily computed using

$$|X(e^{j\omega})|^2 = X(e^{j\omega})X(e^{-j\omega}). \tag{3.22}$$

Table 3.1: Symmetry relations of the discrete-time Fourier transform of a real sequence.

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$
$x_{ev}[n]$	$X_{re}(e^{j\omega})$
$x_{od}[n]$	$jX_{im}(e^{j\omega})$
	$X(e^{j\omega}) = X^*(e^{-j\omega})$
	$X_{re}(e^{j\omega}) = X_{re}(e^{-j\omega})$
Symmetry relations	$X_{im}(e^{j\omega}) = -X_{im}(e^{-j\omega})$
	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$
	$\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

Note: $x_{ev}[n]$ and $x_{od}[n]$ denote the even and odd parts of $x[n]$, respectively.

Table 3.2: Symmetry relations of the discrete-time Fourier transform of a complex sequence.

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$x_{re}[n]$	$X_{cs}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$
$jx_{im}[n]$	$X_{ca}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$
$x_{cs}[n]$	$X_{re}(e^{j\omega})$
$x_{ca}[n]$	$jX_{im}(e^{j\omega})$

Note: $X_{cs}(e^{j\omega})$ and $X_{ca}(e^{j\omega})$ are the conjugate-symmetric and conjugate-antisymmetric parts of $X(e^{j\omega})$, respectively. Likewise, $x_{cs}[n]$ and $x_{ca}[n]$ are the conjugate-symmetric and conjugate-antisymmetric parts of $x[n]$, respectively.

Example 3.7 illustrates the symmetry properties of the Fourier transform of a real sequence.

EXAMPLE 3.7 Real and Imaginary Parts and Magnitude and Phase Functions of a Discrete-Time Fourier Transform

The Fourier transform given by Eq. (3.13) of the real sequence of Eq. (3.12) can be rewritten as

$$X(e^{j\omega}) = \frac{1}{1 - \alpha\, e^{-j\omega}}$$