5 CHAPTER

Conductors and Dielectrics

In this chapter, we apply the methods we have learned to some of the materials with which an engineer must work. In the first part of the chapter, we consider conducting materials by describing the parameters that relate current to an applied electric field. This leads to a general definition of Ohm's law. We then develop methods of evaluating resistances of conductors in a few simple geometric forms. Conditions that must be met at a conducting boundary are obtained next, and this knowledge leads to a discussion of the method of images. The properties of semiconductors are described to conclude the discussion of conducting media.

In the second part of the chapter, we consider insulating materials, or dielectrics. Such materials differ from conductors in that ideally, there is no free charge that can be transported within them to produce conduction current. Instead, all charge is confined to molecular or lattice sites by coulomb forces. An applied electric field has the effect of displacing the charges slightly, leading to the formation of ensembles of electric dipoles. The extent to which this occurs is measured by the relative permittivity, or dielectric constant. Polarization of the medium may modify the electric field, whose magnitude and direction may differ from the values it would have in a different medium or in free space. Boundary conditions for the fields at interfaces between dielectrics are developed to evaluate these differences.

It should be noted that most materials will possess both dielectric and conductive properties; that is, a material considered a dielectric may be slightly conductive, and a material that is mostly conductive may be slightly polarizable. These departures from the ideal cases lead to some interesting behavior, particularly as to the effects on electromagnetic wave propagation, as we will see later. ■

5.1 CURRENT AND CURRENT DENSITY

Electric charges in motion constitute a *current*. The unit of current is the ampere (A), defined as a rate of movement of charge passing a given reference point (or crossing a given reference plane) of one coulomb per second. Current is symbolized by I, and therefore

$$I = \frac{dQ}{dt} \tag{1}$$

Current is thus defined as the motion of positive charges, even though conduction in metals takes place through the motion of electrons, as we will see shortly.

In field theory, we are usually interested in events occurring at a point rather than within a large region, and we find the concept of *current density,* measured in amperes per square meter (A/m^2), more useful. Current density is a vector[1] represented by **J**.

The increment of current ΔI crossing an incremental surface ΔS normal to the current density is

$$\Delta I = J_N \Delta S$$

and in the case where the current density is not perpendicular to the surface,

$$\Delta I = \mathbf{J} \cdot \Delta \mathbf{S}$$

Total current is obtained by integrating,

$$\boxed{I = \int_S \mathbf{J} \cdot d\mathbf{S}} \tag{2}$$

Current density may be related to the velocity of volume charge density at a point. Consider the element of charge $\Delta Q = \rho_\nu \Delta \nu = \rho_\nu \, \Delta S \, \Delta L$, as shown in Figure 5.1*a*. To simplify the explanation, assume that the charge element is oriented with its edges parallel to the coordinate axes and that it has only an x component of velocity. In the time interval Δt, the element of charge has moved a distance Δx, as indicated in Figure 5.1*b*. We have therefore moved a charge $\Delta Q = \rho_\nu \, \Delta S \, \Delta x$ through a reference plane perpendicular to the direction of motion in a time increment Δt, and the resulting current is

$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_\nu \, \Delta S \frac{\Delta x}{\Delta t}$$

As we take the limit with respect to time, we have

$$\Delta I = \rho_\nu \, \Delta S \, v_x$$

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[1] Current is not a vector, for it is easy to visualize a problem in which a total current I in a conductor of nonuniform cross section (such as a sphere) may have a different direction at each point of a given cross section. Current in an exceedingly fine wire, or a *filamentary current*, is occasionally defined as a vector, but we usually prefer to be consistent and give the direction to the filament, or path, and not to the current.

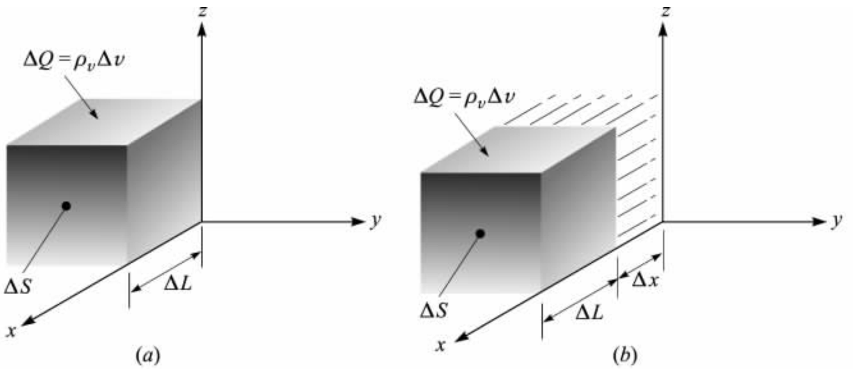

Figure 5.1 An increment of charge, $\Delta Q = \rho_v \Delta S \Delta L$, which moves a distance Δx in a time Δt, produces a component of current density in the limit of $J_x = \rho_v v_x$.

where v_x represents the x component of the velocity $\mathbf{v}$.[2] In terms of current density, we find

$$J_x = \rho_v \, v_x$$

and in general

$$\boxed{\mathbf{J} = \rho_v \mathbf{v}} \tag{3}$$

This last result shows clearly that charge in motion constitutes a current. We call this type of current a *convection current,* and $\mathbf{J}$ or $\rho_v \mathbf{v}$ is the *convection current density*. Note that the convection current density is related linearly to charge density as well as to velocity. The mass rate of flow of cars (cars per square foot per second) in the Holland Tunnel could be increased either by raising the density of cars per cubic foot, or by going to higher speeds, if the drivers were capable of doing so.

D5.1. Given the vector current density $\mathbf{J} = 10\rho^2 z \mathbf{a}_\rho - 4\rho \cos^2 \phi \, \mathbf{a}_\phi$ mA/m^2: (*a*) find the current density at $P(\rho = 3, \phi = 30^\circ, z = 2)$; (*b*) determine the total current flowing outward through the circular band $\rho = 3$, $0 < \phi < 2\pi$, $2 < z < 2.8$.

Ans. $180\mathbf{a}_\rho - 9\mathbf{a}_\phi$ mA/m^2; 3.26 A

5.2 CONTINUITY OF CURRENT

The introduction of the concept of current is logically followed by a discussion of the conservation of charge and the continuity equation. The principle of conservation of charge states simply that charges can be neither created nor destroyed, although equal

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[2] The lowercase v is used both for volume and velocity. Note, however, that velocity always appears as a vector $\mathbf{v}$, a component v_x, or a magnitude $|\mathbf{v}|$, whereas volume appears only in differential form as dv or Δv.

amounts of positive and negative charge may be *simultaneously* created, obtained by separation, or lost by recombination.

The continuity equation follows from this principle when we consider any region bounded by a closed surface. The current through the closed surface is

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S}$$

and this *outward flow* of positive charge must be balanced by a decrease of positive charge (or perhaps an increase of negative charge) within the closed surface. If the charge inside the closed surface is denoted by Q_i, then the rate of decrease is $-dQ_i/dt$ and the principle of conservation of charge requires

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{dQ_i}{dt} \tag{4}$$

It might be well to answer here an often-asked question. "Isn't there a sign error? I thought $I = dQ/dt$." The presence or absence of a negative sign depends on what current and charge we consider. In circuit theory we usually associate the current flow *into* one terminal of a capacitor with the time rate of increase of charge on that plate. The current of (4), however, is an *outward-flowing* current.

Equation (4) is the integral form of the continuity equation; the differential, or point, form is obtained by using the divergence theorem to change the surface integral into a volume integral:

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = \int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv$$

We next represent the enclosed charge Q_i by the volume integral of the charge density,

$$\int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv = -\frac{d}{dt} \int_{\text{vol}} \rho_v\, dv$$

If we agree to keep the surface constant, the derivative becomes a partial derivative and may appear within the integral,

$$\int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv = \int_{\text{vol}} -\frac{\partial \rho_v}{\partial t}\, dv$$

from which we have our point form of the continuity equation,

$$\boxed{(\nabla \cdot \mathbf{J}) = -\frac{\partial \rho_v}{\partial t}} \tag{5}$$

Remembering the physical interpretation of divergence, this equation indicates that the current, or charge per second, diverging from a small volume per unit volume is equal to the time rate of decrease of charge per unit volume at every point.

As a numerical example illustrating some of the concepts from the last two sections, let us consider a current density that is directed radially outward and decreases exponentially with time,

$$\mathbf{J} = \frac{1}{r} e^{-t} \mathbf{a}_r \text{ A/m}^2$$

Selecting an instant of time $t = 1$ s, we may calculate the total outward current at $r = 5$ m:

$$I = J_r S = \left(\tfrac{1}{5}e^{-1}\right)(4\pi 5^2) = 23.1 \text{ A}$$

At the same instant, but for a slightly larger radius, $r = 6$ m, we have

$$I = J_r S = \left(\tfrac{1}{6}e^{-1}\right)\left(4\pi 6^2\right) = 27.7 \text{ A}$$

Thus, the total current is larger at $r = 6$ than it is at $r = 5$.

To see why this happens, we need to look at the volume charge density and the velocity. We use the continuity equation first:

$$-\frac{\partial \rho_v}{\partial t} = \nabla \cdot \mathbf{J} = \nabla \cdot \left(\frac{1}{r}e^{-t}\mathbf{a}_r\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r}e^{-t}\right) = \frac{1}{r^2}e^{-t}$$

We next seek the volume charge density by integrating with respect to t. Because ρ_v is given by a partial derivative with respect to time, the "constant" of integration may be a function of r:

$$\rho_v = -\int \frac{1}{r^2}e^{-t}\,dt + K(r) = \frac{1}{r^2}e^{-t} + K(r)$$

If we assume that $\rho_v \to 0$ as $t \to \infty$, then $K(r) = 0$, and

$$\rho_v = \frac{1}{r^2}e^{-t} \text{ C/m}^3$$

We may now use $\mathbf{J} = \rho_v \mathbf{v}$ to find the velocity,

$$v_r = \frac{J_r}{\rho_v} = \frac{\dfrac{1}{r}e^{-t}}{\dfrac{1}{r^2}e^{-t}} = r \text{ m/s}$$

The velocity is greater at $r = 6$ than it is at $r = 5$, and we see that some (unspecified) force is accelerating the charge density in an outward direction.

In summary, we have a current density that is inversely proportional to r, a charge density that is inversely proportional to r^2, and a velocity and total current that are proportional to r. All quantities vary as e^{-t}.

D5.2. Current density is given in cylindrical coordinates as $\mathbf{J} = -10^6 z^{1.5}\mathbf{a}_z$ A/m^2 in the region $0 \le \rho \le 20\,\mu$m; for $\rho \ge 20\,\mu$m, $\mathbf{J} = 0$. (*a*) Find the total current crossing the surface $z = 0.1$ m in the $\mathbf{a}_z$ direction. (*b*) If the charge velocity is 2×10^6 m/s at $z = 0.1$ m, find ρ_v there. (*c*) If the volume charge density at $z = 0.15$ m is -2000 C/m^3, find the charge velocity there.

Ans. $-39.7\,\mu$A; -15.8 mC/m^3; 29.0 m/s

5.3 METALLIC CONDUCTORS

Physicists describe the behavior of the electrons surrounding the positive atomic nucleus in terms of the total energy of the electron with respect to a zero reference level for an electron at an infinite distance from the nucleus. The total energy is the sum of the kinetic and potential energies, and because energy must be given to an electron to pull it away from the nucleus, the energy of every electron in the atom is a negative quantity. Even though this picture has some limitations, it is convenient to associate these energy values with orbits surrounding the nucleus, the more negative energies corresponding to orbits of smaller radius. According to the quantum theory, only certain discrete energy levels, or energy states, are permissible in a given atom, and an electron must therefore absorb or emit discrete amounts of energy, or quanta, in passing from one level to another. A normal atom at absolute zero temperature has an electron occupying every one of the lower energy shells, starting outward from the nucleus and continuing until the supply of electrons is exhausted.

In a crystalline solid, such as a metal or a diamond, atoms are packed closely together, many more electrons are present, and many more permissible energy levels are available because of the interaction forces between adjacent atoms. We find that the allowed energies of electrons are grouped into broad ranges, or "bands," each band consisting of very numerous, closely spaced, discrete levels. At a temperature of absolute zero, the normal solid also has every level occupied, starting with the lowest and proceeding in order until all the electrons are located. The electrons with the highest (least negative) energy levels, the valence electrons, are located in the *valence band*. If there are permissible higher-energy levels in the valence band, or if the valence band merges smoothly into a *conduction band*, then additional kinetic energy may be given to the valence electrons by an external field, resulting in an electron flow. The solid is called a *metallic conductor.* The filled valence band and the unfilled conduction band for a conductor at absolute zero temperature are suggested by the sketch in Figure 5.2*a*.

If, however, the electron with the greatest energy occupies the top level in the valence band and a gap exists between the valence band and the conduction band, then

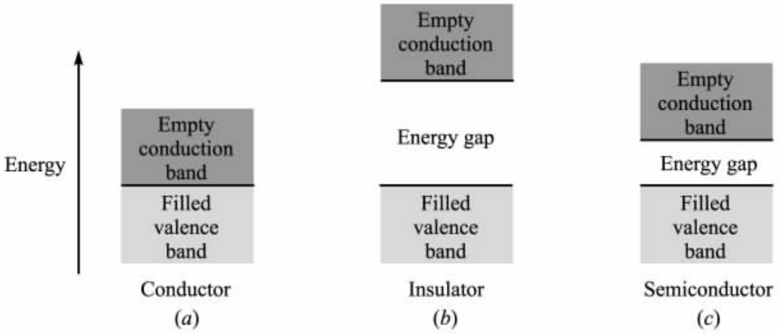

Figure 5.2 The energy-band structure in three different types of materials at 0 K. (*a*) The conductor exhibits no energy gap between the valence and conduction bands. (*b*) The insulator shows a large energy gap. (*c*) The semiconductor has only a small energy gap.

the electron cannot accept additional energy in small amounts, and the material is an insulator. This band structure is indicated in Figure 5.2*b*. Note that if a relatively large amount of energy can be transferred to the electron, it may be sufficiently excited to jump the gap into the next band where conduction can occur easily. Here the insulator breaks down.

An intermediate condition occurs when only a small "forbidden region" separates the two bands, as illustrated by Figure 5.2*c*. Small amounts of energy in the form of heat, light, or an electric field may raise the energy of the electrons at the top of the filled band and provide the basis for conduction. These materials are insulators which display many of the properties of conductors and are called *semiconductors*.

Let us first consider the conductor. Here the valence electrons, or *conduction,* or *free,* electrons, move under the influence of an electric field. With a field **E**, an electron having a charge $Q = -e$ will experience a force

$$\mathbf{F} = -e\mathbf{E}$$

In free space, the electron would accelerate and continuously increase its velocity (and energy); in the crystalline material, the progress of the electron is impeded by continual collisions with the thermally excited crystalline lattice structure, and a constant average velocity is soon attained. This velocity $\mathbf{v}_d$ is termed the *drift velocity,* and it is linearly related to the electric field intensity by the *mobility* of the electron in the given material. We designate mobility by the symbol μ (mu), so that

$$\mathbf{v}_d = -\mu_e \mathbf{E} \tag{6}$$

where μ_ϵ is the mobility of an electron and is positive by definition. Note that the electron velocity is in a direction opposite to the direction of **E**. Equation (6) also shows that mobility is measured in the units of square meters per volt-second; typical values[3] are 0.0012 for aluminum, 0.0032 for copper, and 0.0056 for silver.

For these good conductors, a drift velocity of a few centimeters per second is sufficient to produce a noticeable temperature rise and can cause the wire to melt if the heat cannot be quickly removed by thermal conduction or radiation.

Substituting (6) into Eq. (3) of Section 5.1, we obtain

$$\boxed{\mathbf{J} = -\rho_e \mu_e \mathbf{E}} \tag{7}$$

where ρ_e is the free-electron charge density, a negative value. The total charge density ρ_v is zero because equal positive and negative charges are present in the neutral material. The negative value of ρ_e and the minus sign lead to a current density **J** that is in the same direction as the electric field intensity **E**.

The relationship between **J** and **E** for a metallic conductor, however, is also specified by the conductivity σ (sigma),

$$\boxed{\mathbf{J} = \sigma \mathbf{E}} \tag{8}$$

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[3] Wert and Thomson, p. 238, listed in the References at the end of this chapter.

where σ is measured is siemens[4] per meter (S/m). One siemens (1 S) is the basic unit of conductance in the SI system and is defined as one ampere per volt. Formerly, the unit of conductance was called the mho and was symbolized by an *inverted* Ω. Just as the siemens honors the Siemens brothers, the reciprocal unit of resistance that we call the ohm (1 Ω is one volt per ampere) honors Georg Simon Ohm, a German physicist who first described the current-voltage relationship implied by Eq. (8). We call this equation the *point form of Ohm's law;* we will look at the more common form of Ohm's law shortly.

First, however, it is informative to note the conductivity of several metallic conductors; typical values (in siemens per meter) are 3.82×10^7 for aluminum, 5.80×10^7 for copper, and 6.17×10^7 for silver. Data for other conductors may be found in Appendix C. On seeing data such as these, it is only natural to assume that we are being presented with *constant* values; this is essentially true. Metallic conductors obey Ohm's law quite faithfully, and it is a *linear* relationship; the conductivity is constant over wide ranges of current density and electric field intensity. Ohm's law and the metallic conductors are also described as *isotropic,* or having the same properties in every direction. A material which is not isotropic is called *anisotropic,* and we shall mention such a material in Chapter 6.

The conductivity is a function of temperature, however. The resistivity, which is the reciprocal of the conductivity, varies almost linearly with temperature in the region of room temperature, and for aluminum, copper, and silver it increases about 0.4 percent for a 1-K rise in temperature.[5] For several metals the resistivity drops abruptly to zero at a temperature of a few kelvin; this property is termed *superconductivity*. Copper and silver are not superconductors, although aluminum is (for temperatures below 1.14 K).

If we now combine Equations (7) and (8), conductivity may be expressed in terms of the charge density and the electron mobility,

$$\boxed{\sigma = -\rho_e \mu_e} \tag{9}$$

From the definition of mobility (6), it is now satisfying to note that a higher temperature infers a greater crystalline lattice vibration, more impeded electron progress for a given electric field strength, lower drift velocity, lower mobility, lower conductivity from Eq. (9), and higher resistivity as stated.

The application of Ohm's law in point form to a macroscopic (visible to the naked eye) region leads to a more familiar form. Initially, assume that **J** and **E** are *uniform,* as they are in the cylindrical region shown in Figure 5.3. Because they are uniform,

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} = JS \tag{10}$$

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[4] This is the family name of two German-born brothers, Karl Wilhelm and Werner von Siemens, who were famous engineer-inventors in the nineteenth century. Karl became a British subject and was knighted, becoming Sir William Siemens.

[5] Copious temperature data for conducting materials are available in the *Standard Handbook for Electrical Engineers,* listed among the References at the end of this chapter.

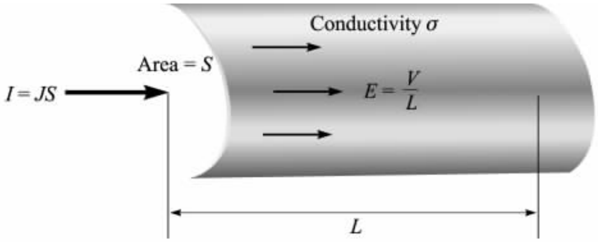

Figure 5.3 Uniform current density J and electric field intensity E in a cylindrical region of length L and cross-sectional area S. Here $V = IR$, where $R = L/\sigma S$.

and

$$V_{ab} = -\int_b^a \mathbf{E} \cdot d\mathbf{L} = -\mathbf{E} \cdot \int_b^a d\mathbf{L} = -\mathbf{E} \cdot \mathbf{L}_{ba}$$
$$= \mathbf{E} \cdot \mathbf{L}_{ab} \tag{11}$$

or

$$V = EL$$

Thus

$$J = \frac{I}{S} = \sigma E = \sigma \frac{V}{L}$$

or

$$V = \frac{L}{\sigma S} I$$

The ratio of the potential difference between the two ends of the cylinder to the current entering the more positive end, however, is recognized from elementary circuit theory as the *resistance* of the cylinder, and therefore

$$\boxed{V = IR} \tag{12}$$

where

$$\boxed{R = \frac{L}{\sigma S}} \tag{13}$$

Equation (12) is, of course, known as *Ohm's law*, and Eq. (13) enables us to compute the resistance R, measured in ohms (abbreviated as Ω), of conducting objects which possess uniform fields. If the fields are not uniform, the resistance may still be defined as the ratio of V to I, where V is the potential difference between two specified equipotential surfaces in the material and I is the total current crossing the more positive surface into the material. From the general integral relationships in Eqs. (10) and (11), and from Ohm's law (8), we may write this general expression for resistance when the fields are nonuniform,

$$R = \frac{V_{ab}}{I} = \frac{-\int_b^a \mathbf{E} \cdot d\mathbf{L}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{S}} \tag{14}$$

The line integral is taken between two equipotential surfaces in the conductor, and the surface integral is evaluated over the more positive of these two equipotentials. We cannot solve these nonuniform problems at this time, but we should be able to solve several of them after reading Chapter 6.

EXAMPLE 5.1

As an example of the determination of the resistance of a cylinder, we find the resistance of a 1-mile length of #16 copper wire, which has a diameter of 0.0508 in.

Solution. The diameter of the wire is $0.0508 \times 0.0254 = 1.291 \times 10^{-3}$ m, the area of the cross section is $\pi(1.291 \times 10^{-3}/2)^2 = 1.308 \times 10^{-6}$ m^2, and the length is 1609 m. Using a conductivity of 5.80×10^7 S/m, the resistance of the wire is, therefore,

$$R = \frac{1609}{(5.80 \times 10^7)(1.308 \times 10^{-6})} = 21.2\ \Omega$$

This wire can safely carry about 10 A dc, corresponding to a current density of $10/(1.308 \times 10^{-6}) = 7.65 \times 10^6$ A/m^2, or 7.65 A/mm^2. With this current, the potential difference between the two ends of the wire is 212 V, the electric field intensity is 0.312 V/m, the drift velocity is 0.000 422 m/s, or a little more than one furlong a week, and the free-electron charge density is -1.81×10^{10} C/m^3, or about one electron within a cube two angstroms on a side.

D5.3. Find the magnitude of the current density in a sample of silver for which $\sigma = 6.17 \times 10^7$ S/m and $\mu_e = 0.0056\,\text{m}^2/\text{V}\cdot\text{s}$ if (*a*) the drift velocity is 1.5 μm/s ; (*b*) the electric field intensity is 1 mV/m; (*c*) the sample is a cube 2.5 mm on a side having a voltage of 0.4 mV between opposite faces; (*d*) the sample is a cube 2.5 mm on a side carrying a total current of 0.5 A.

Ans. 16.5 kA/m^2; 61.7 kA/m^2; 9.9 MA/m^2; 80.0 kA/m^2

D5.4. A copper conductor has a diameter of 0.6 in. and it is 1200 ft long. Assume that it carries a total dc current of 50 A. (*a*) Find the total resistance of the conductor. (*b*) What current density exists in it? (*c*) What is the dc voltage between the conductor ends? (*d*) How much power is dissipated in the wire?

Ans. 0.035 Ω; 2.74×10^5 A/m^2; 1.73 V; 86.4 W

5.4 CONDUCTOR PROPERTIES AND BOUNDARY CONDITIONS

Once again, we must temporarily depart from our assumed static conditions and let time vary for a few microseconds to see what happens when the charge distribution is suddenly unbalanced within a conducting material. Suppose, for the sake of argument, that there suddenly appear a number of electrons in the interior of a conductor. The electric fields set up by these electrons are not counteracted by any positive charges, and the electrons therefore begin to accelerate away from each other. This continues until the electrons reach the surface of the conductor or until a number of electrons equal to the number injected have reached the surface.

Here, the outward progress of the electrons is stopped, for the material surrounding the conductor is an insulator not possessing a convenient conduction band. No charge may remain within the conductor. If it did, the resulting electric field would force the charges to the surface.

Hence the final result within a conductor is zero charge density, and a surface charge density resides on the exterior surface. This is one of the two characteristics of a good conductor.

The other characteristic, stated for static conditions in which no current may flow, follows directly from Ohm's law: the electric field intensity within the conductor is zero. Physically, we see that if an electric field were present, the conduction electrons would move and produce a current, thus leading to a nonstatic condition.

Summarizing for electrostatics, no charge and no electric field may exist at any point *within* a conducting material. Charge may, however, appear on the surface as a surface charge density, and our next investigation concerns the fields *external* to the conductor.

We wish to relate these external fields to the charge on the surface of the conductor. The problem is a simple one, and we may first talk our way to the solution with a little mathematics.

If the external electric field intensity is decomposed into two components, one tangential and one normal to the conductor surface, the tangential component is seen to be zero. If it were not zero, a tangential force would be applied to the elements of the surface charge, resulting in their motion and nonstatic conditions. Because static conditions are assumed, the tangential electric field intensity and electric flux density are zero.

Gauss's law answers our questions concerning the normal component. The electric flux leaving a small increment of surface must be equal to the charge residing on that incremental surface. The flux cannot penetrate into the conductor, for the total field there is zero. It must then leave the surface normally. Quantitatively, we may say that the electric flux density in coulombs per square meter leaving the surface normally is equal to the surface charge density in coulombs per square meter, or $D_N = \rho_S$.

If we use some of our previously derived results in making a more careful analysis (and incidentally introducing a general method which we must use later), we should set up a boundary between a conductor and free space (Figure 5.4) showing tangential

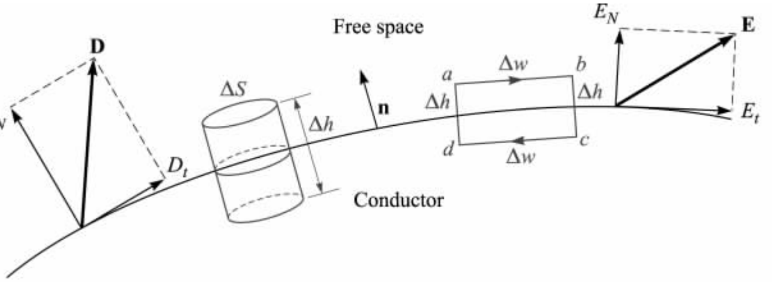

Figure 5.4 An appropriate closed path and gaussian surface are used to determine boundary conditions at a boundary between a conductor and free space; $E_t = 0$ and $D_N = \rho_S$.

and normal components of **D** and **E** on the free-space side of the boundary. Both fields are zero in the conductor. The tangential field may be determined by applying Section 4.5, Eq. (21),

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

around the small closed path *abcda*. The integral must be broken up into four parts

$$\int_a^b + \int_b^c + \int_c^d + \int_d^a = 0$$

Remembering that $\mathbf{E} = 0$ within the conductor, we let the length from a to b or c to d be Δw and from b to c or d to a be Δh, and obtain

$$E_t \Delta w - E_{N,\text{at}\, b} \tfrac{1}{2} \Delta h + E_{N,\text{at}\, a} \tfrac{1}{2} \Delta h = 0$$

As we allow Δh to approach zero, keeping Δw small but finite, it makes no difference whether or not the normal fields are equal at a and b, for Δh causes these products to become negligibly small. Hence, $E_t \Delta w = 0$ and, therefore, $E_t = 0$.

The condition on the normal field is found most readily by considering D_N rather than E_N and choosing a small cylinder as the gaussian surface. Let the height be Δh and the area of the top and bottom faces be ΔS. Again, we let Δh approach zero. Using Gauss's law,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

we integrate over the three distinct surfaces

$$\int_{\text{top}} + \int_{\text{bottom}} + \int_{\text{sides}} = Q$$

and find that the last two are zero (for different reasons). Then

$$D_N \Delta S = Q = \rho_S \Delta S$$

or

$$D_N = \rho_S$$

These are the desired *boundary conditions* for the conductor-to-free-space boundary in electrostatics,

$$\boxed{D_t = E_t = 0} \tag{15}$$

$$\boxed{D_N = \epsilon_0 E_N = \rho_S} \tag{16}$$

The electric flux leaves the conductor in a direction normal to the surface, and the value of the electric flux density is numerically equal to the surface charge density. Equations (15) and (16) can be more formally expressed using the vector fields

$$\mathbf{E} \times \mathbf{n}\big|_s = 0 \tag{17}$$

$$\mathbf{D} \cdot \mathbf{n}\big|_s = \rho_s \tag{18}$$

where **n** is the unit normal vector at the surface that points *away* from the conductor, as shown in Figure 5.4, and where both operations are evaluated at the conductor surface, s. Taking the cross product or the dot product of either field quantity with **n** gives the tangential or the normal component of the field, respectively.

An immediate and important consequence of a zero tangential electric field intensity is the fact that a conductor surface is an equipotential surface. The evaluation of the potential difference between any two points on the surface by the line integral leads to a zero result, because the path may be chosen on the surface itself where $\mathbf{E} \cdot d\mathbf{L} = 0$.

To summarize the principles which apply to conductors in electrostatic fields, we may state that

1. The static electric field intensity inside a conductor is zero.
2. The static electric field intensity at the surface of a conductor is everywhere directed normal to that surface.
3. The conductor surface is an equipotential surface.

Using these three principles, there are a number of quantities that may be calculated at a conductor boundary, given a knowledge of the potential field.

EXAMPLE 5.2

Given the potential,

$$V = 100(x^2 - y^2)$$

and a point $P(2, -1, 3)$ that is stipulated to lie on a conductor-to-free-space boundary, find V, **E**, **D**, and ρ_S at P, and also the equation of the conductor surface.

Solution. The potential at point P is

$$V_P = 100[2^2 - (-1)^2] = 300 \text{ V}$$

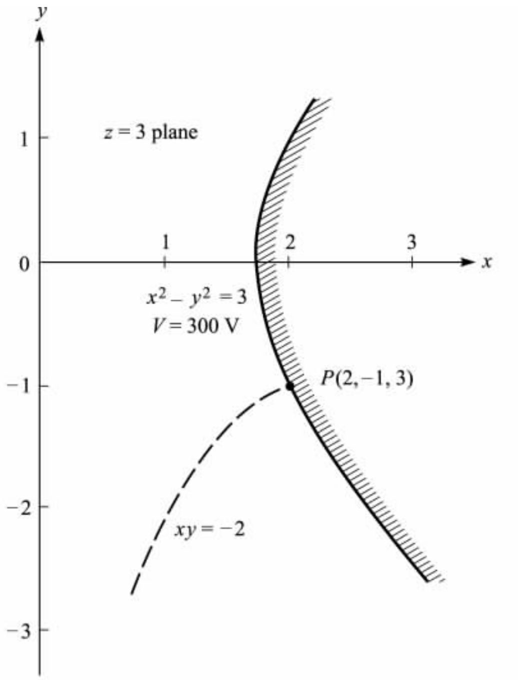

Figure 5.5 Given point $P(2, -1, 3)$ and the potential field, $V = 100(x^2 - y^2)$, we find the equipotential surface through P is $x^2 - y^2 = 3$, and the streamline through P is $xy = -2$.

Because the conductor is an equipotential surface, the potential at the entire surface must be 300 V. Moreover, if the conductor is a solid object, then the potential everywhere in and on the conductor is 300 V, for $\mathbf{E} = 0$ within the conductor.

The equation representing the locus of all points having a potential of 300 V is

$$300 = 100(x^2 - y^2)$$

or

$$x^2 - y^2 = 3$$

This is therefore the equation of the conductor surface; it happens to be a hyperbolic cylinder, as shown in Figure 5.5. Let us assume arbitrarily that the solid conductor lies above and to the right of the equipotential surface at point P, whereas free space is down and to the left.

Next, we find $\mathbf{E}$ by the gradient operation,

$$\mathbf{E} = -100\nabla(x^2 - y^2) = -200x\mathbf{a}_x + 200y\mathbf{a}_y$$

At point P,

$$\mathbf{E}_p = -400\mathbf{a}_x - 200\mathbf{a}_y \text{ V/m}$$

Because $\mathbf{D} = \epsilon_0 \mathbf{E}$, we have

$$\mathbf{D}_P = 8.854 \times 10^{-12} \mathbf{E}_P = -3.54\mathbf{a}_x - 1.771\mathbf{a}_y \ \text{nC/m}^2$$

The field is directed downward and to the left at P; it is normal to the equipotential surface. Therefore,

$$D_N = |\mathbf{D}_P| = 3.96 \ \text{nC/m}^2$$

Thus, the surface charge density at P is

$$\rho_{S,P} = D_N = 3.96 \ \text{nC/m}^2$$

Note that if we had taken the region to the left of the equipotential surface as the conductor, the **E** field would *terminate* on the surface charge and we would let $\rho_S = -3.96 \ \text{nC/m}^2$.

EXAMPLE 5.3

Finally, let us determine the equation of the streamline passing through P.

Solution. We see that

$$\frac{E_y}{E_x} = \frac{200y}{-200x} = -\frac{y}{x} = \frac{dy}{dx}$$

Thus,

$$\frac{dy}{y} + \frac{dx}{x} = 0$$

and

$$\ln y + \ln x = C_1$$

Therefore,

$$xy = C_2$$

The line (or surface) through P is obtained when $C_2 = (2)(-1) = -2$. Thus, the streamline is the trace of another hyperbolic cylinder,

$$xy = -2$$

This is also shown on Figure 5.5.

D5.5. Given the potential field in free space, $V = 100 \sinh 5x \sin 5y \ V$, and a point $P(0.1, 0.2, 0.3)$, find at P: (*a*) V; (*b*) $\mathbf{E}$; (*c*) $|\mathbf{E}|$; (*d*) $|\rho_S|$ if it is known that P lies on a conductor surface.

Ans. 43.8 V; $-474\mathbf{a}_x - 140.8\mathbf{a}_y$ V/m; 495 V/m; 4.38 nC/m^2

5.5 THE METHOD OF IMAGES

One important characteristic of the dipole field that we developed in Chapter 4 is the infinite plane at zero potential that exists midway between the two charges. Such a plane may be represented by a vanishingly thin conducting plane that is infinite in extent. The conductor is an equipotential surface at a potential $V = 0$, and the electric field intensity is therefore normal to the surface. Thus, if we replace the dipole configuration shown in Figure 5.6*a* with the single charge and conducting plane shown in Figure 5.6*b*, the fields in the upper half of each figure are the same. Below the conducting plane, all fields are zero, as we have not provided any charges in that region. Of course, we might also substitute a single negative charge below a conducting plane for the dipole arrangement and obtain equivalence for the fields in the lower half of each region.

If we approach this equivalence from the opposite point of view, we begin with a single charge above a perfectly conducting plane and then see that we may maintain the same fields above the plane by removing the plane and locating a negative charge at a symmetrical location below the plane. This charge is called the *image* of the original charge, and it is the negative of that value.

If we can do this once, linearity allows us to do it again and again, and thus *any* charge configuration above an infinite ground plane may be replaced by an arrangement composed of the given charge configuration, its image, and no conducting plane. This is suggested by the two illustrations of Figure 5.7. In many cases, the potential field of the new system is much easier to find since it does not contain the conducting plane with its unknown surface charge distribution.

As an example of the use of images, let us find the surface charge density at $P(2, 5, 0)$ on the conducting plane $z = 0$ if there is a line charge of 30 nC/m located at $x = 0$, $z = 3$, as shown in Figure 5.8*a*. We remove the plane and install an image line charge of -30 nC/m at $x = 0$, $z = -3$, as illustrated in Figure 5.8*b*. The field at P may now be obtained by superposition of the known fields of the line

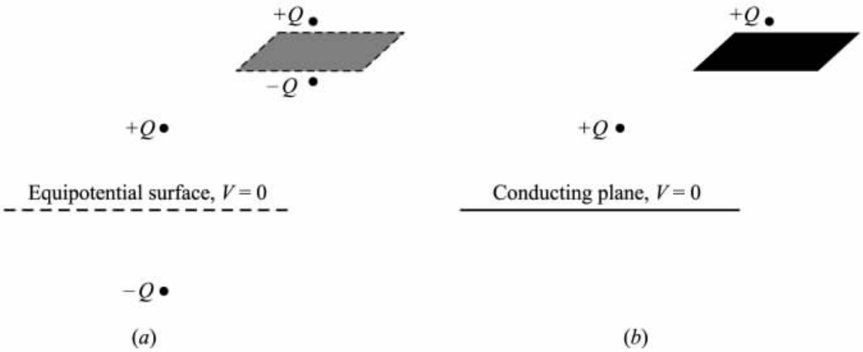

Figure 5.6 (*a*) Two equal but opposite charges may be replaced by (*b*) a single charge and a conducting plane without affecting the fields above the $V = 0$ surface.

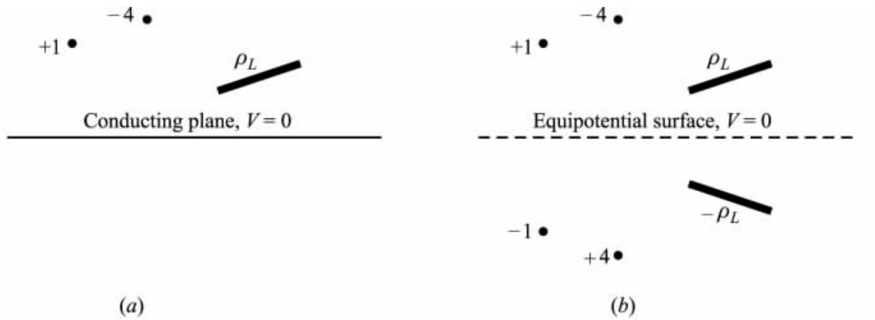

Figure 5.7 (*a*) A given charge configuration above an infinite conducting plane may be replaced by (*b*) the given charge configuration plus the image configuration, without the conducting plane.

charges. The radial vector from the positive line charge to P is $\mathbf{R}_+ = 2\mathbf{a}_x - 3\mathbf{a}_z$, while $\mathbf{R}_- = 2\mathbf{a}_x + 3\mathbf{a}_z$. Thus, the individual fields are

$$\mathbf{E}_+ = \frac{\rho_L}{2\pi\epsilon_0 R_+}\mathbf{a}_{R+} = \frac{30\times 10^{-9}}{2\pi\epsilon_0\sqrt{13}}\,\frac{2\mathbf{a}_x - 3\mathbf{a}_z}{\sqrt{13}}$$

and

$$\mathbf{E}_- = \frac{30\times 10^{-9}}{2\pi\epsilon_0\sqrt{13}}\,\frac{2\mathbf{a}_x + 3\mathbf{a}_z}{\sqrt{13}}$$

Adding these results, we have

$$\mathbf{E} = \frac{-180\times 10^{-9}\mathbf{a}_z}{2\pi\epsilon_0(13)} = -249\mathbf{a}_z \text{ V/m}$$

This then is the field at (or just above) P in both the configurations of Figure 5.8, and it is certainly satisfying to note that the field is normal to the conducting plane, as it must be. Thus, $\mathbf{D} = \epsilon_0\mathbf{E} = -2.20\mathbf{a}_z$ nC/m^2, and because this is directed *toward* the conducting plane, ρ_S is negative and has a value of -2.20 nC/m^2 at P.

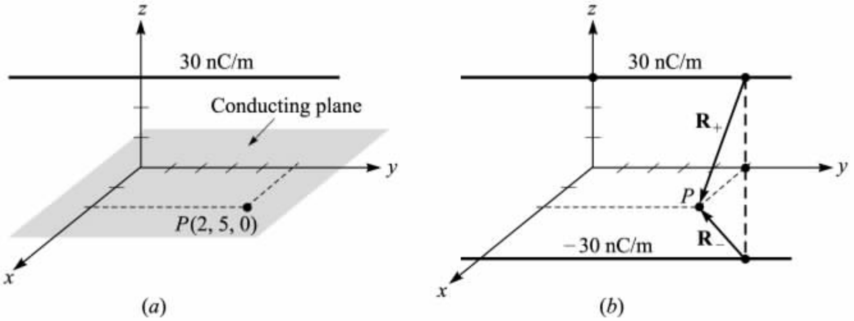

Figure 5.8 (*a*) A line charge above a conducting plane. (*b*) The conductor is removed, and the image of the line charge is added.

D5.6. A perfectly conducting plane is located in free space at $x = 4$, and a uniform infinite line charge of 40 nC/m lies along the line $x = 6$, $y = 3$. Let $V = 0$ at the conducting plane. At $P(7, -1, 5)$ find: (*a*) V; (*b*) $\mathbf{E}$.

Ans. 317 V; $-45.3\mathbf{a}_x - 99.2\mathbf{a}_y$ V/m

5.6 SEMICONDUCTORS

If we now turn our attention to an intrinsic semiconductor material, such as pure germanium or silicon, two types of current carriers are present, electrons and holes. The electrons are those from the top of the filled valence band that have received sufficient energy (usually thermal) to cross the relatively small forbidden band into the conduction band. The forbidden-band energy gap in typical semiconductors is of the order of one electronvolt. The vacancies left by these electrons represent unfilled energy states in the valence band which may also move from atom to atom in the crystal. The vacancy is called a hole, and many semiconductor properties may be described by treating the hole as if it had a positive charge of e, a mobility, μ_h, and an effective mass comparable to that of the electron. Both carriers move in an electric field, and they move in opposite directions; hence each contributes a component of the total current which is in the same direction as that provided by the other. The conductivity is therefore a function of both hole and electron concentrations and mobilities,

$$\boxed{\sigma = -\rho_e \mu_e + \rho_h \mu_h} \tag{19}$$

For pure, or *intrinsic,* silicon, the electron and hole mobilities are 0.12 and 0.025, respectively, whereas for germanium, the mobilities are, respectively, 0.36 and 0.17. These values are given in square meters per volt-second and range from 10 to 100 times as large as those for aluminum, copper, silver, and other metallic conductors.[6] These mobilities are given for a temperature of 300 K.

The electron and hole concentrations depend strongly on temperature. At 300 K the electron and hole volume charge densities are both 0.0024 C/m^{3}in magnitude in intrinsic silicon and 3.0 C/m^3 in intrinsic germanium. These values lead to conductivities of 0.000 35 S/m in silicon and 1.6 S/m in germanium. As temperature increases, the mobilities decrease, but the charge densities increase very rapidly. As a result, the conductivity of silicon increases by a factor of 10 as the temperature increases from 300 to about 330 K and decreases by a factor of 10 as the temperature drops from 300 to about 275 K. Note that the conductivity of the intrinsic semiconductor increases with temperature, whereas that of a metallic conductor decreases with temperature; this is one of the characteristic differences between the metallic conductors and the intrinsic semiconductors.

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[6] Mobility values for semiconductors are given in References 2, 3, and 5 listed at the end of this chapter.

Intrinsic semiconductors also satisfy the point form of Ohm's law; that is, the conductivity is reasonably constant with current density and with the direction of the current density.

The number of charge carriers and the conductivity may both be increased dramatically by adding very small amounts of impurities. *Donor* materials provide additional electrons and form *n-type* semiconductors, whereas *acceptors* furnish extra holes and form *p-type* materials. The process is known as *doping,* and a donor concentration in silicon as low as one part in 10^7 causes an increase in conductivity by a factor of 10^5.

The range of value of the conductivity is extreme as we go from the best insulating materials to semiconductors and the finest conductors. In siemens per meter, σ ranges from 10^{-17} for fused quartz, 10^{-7} for poor plastic insulators, and roughly unity for semiconductors to almost 10^8 for metallic conductors at room temperature. These values cover the remarkably large range of some 25 orders of magnitude.

WW Quizzes

D5.7. Using the values given in this section for the electron and hole mobilities in silicon at 300 K, and assuming hole and electron charge densities are 0.0029 C/m^3 and -0.0029 C/m^3, respectively, find: (*a*) the component of the conductivity due to holes; (*b*) the component of the conductivity due to electrons; (*c*) the conductivity.

Ans. 72.5 μS/m; 348 μS/m; 421 μS/m

5.7 THE NATURE OF DIELECTRIC MATERIALS

WW Interactives

A dielectric in an electric field can be viewed as a free-space arrangement of microscopic electric dipoles, each of which is composed of a positive and a negative charge whose centers do not quite coincide. These are not free charges, and they cannot contribute to the conduction process. Rather, they are bound in place by atomic and molecular forces and can only shift positions slightly in response to external fields. They are called *bound* charges, in contrast to the free charges that determine conductivity. The bound charges can be treated as any other sources of the electrostatic field. Therefore, we would not need to introduce the dielectric constant as a new parameter or to deal with permittivities different from the permittivity of free space; however, the alternative would be to consider *every charge within a piece of dielectric material*. This is too great a price to pay for using all our previous equations in an unmodified form, and we shall therefore spend some time theorizing about dielectrics in a qualitative way; introducing polarization **P**, permittivity ϵ, and relative permittivity ϵ_r; and developing some quantitative relationships involving these new parameters.

The characteristic that all dielectric materials have in common, whether they are solid, liquid, or gas, and whether or not they are crystalline in nature, is their ability to store electric energy. This storage takes place by means of a shift in the relative positions of the internal, bound positive and negative charges against the normal molecular and atomic forces.

This displacement against a restraining force is analogous to lifting a weight or stretching a spring and represents potential energy. The source of the energy is the external field, the motion of the shifting charges resulting perhaps in a transient current through a battery that is producing the field.

The actual mechanism of the charge displacement differs in the various dielectric materials. Some molecules, termed *polar* molecules, have a permanent displacement existing between the centers of "gravity" of the positive and negative charges, and each pair of charges acts as a dipole. Normally the dipoles are oriented in a random way throughout the interior of the material, and the action of the external field is to align these molecules, to some extent, in the same direction. A sufficiently strong field may even produce an additional displacement between the positive and negative charges.

A *nonpolar* molecule does not have this dipole arrangement until after a field is applied. The negative and positive charges shift in opposite directions against their mutual attraction and produce a dipole that is aligned with the electric field.

Either type of dipole may be described by its dipole moment **p**, as developed in Section 4.7, Eq. (36),

$$\mathbf{p} = Q\mathbf{d} \tag{20}$$

where Q is the positive one of the two bound charges composing the dipole, and **d** is the vector from the negative to the positive charge. We note again that the units of **p** are coulomb-meters.

If there are n dipoles per unit volume and we deal with a volume Δv, then there are $n\,\Delta v$ dipoles, and the total dipole moment is obtained by the vector sum,

$$\mathbf{p}_{\text{total}} = \sum_{i=1}^{n\,\Delta v} \mathbf{p}_i$$

If the dipoles are aligned in the same general direction, $\mathbf{p}_{\text{total}}$ may have a significant value. However, a random orientation may cause $\mathbf{p}_{\text{total}}$ to be essentially zero.

We now define the polarization **P** as the *dipole moment per unit volume,*

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\,\Delta v} \mathbf{p}_i \tag{21}$$

with units of coulombs per square meter. We will treat **P** as a typical continuous field, even though it is obvious that it is essentially undefined at points within an atom or molecule. Instead, we should think of its value at any point as an average value taken over a sample volume Δv—large enough to contain many molecules ($n\,\Delta v$ in number), but yet sufficiently small to be considered incremental in concept.

Our immediate goal is to show that the bound-volume charge density acts like the free-volume charge density in producing an external field; we will obtain a result similar to Gauss's law.

To be specific, assume that we have a dielectric containing nonpolar molecules. No molecule has a dipole moment, and $\mathbf{P} = 0$ throughout the material. Somewhere in the interior of the dielectric we select an incremental surface element $\Delta\mathbf{S}$, as shown in Figure 5.9*a*, and apply an electric field **E**. The electric field produces a moment

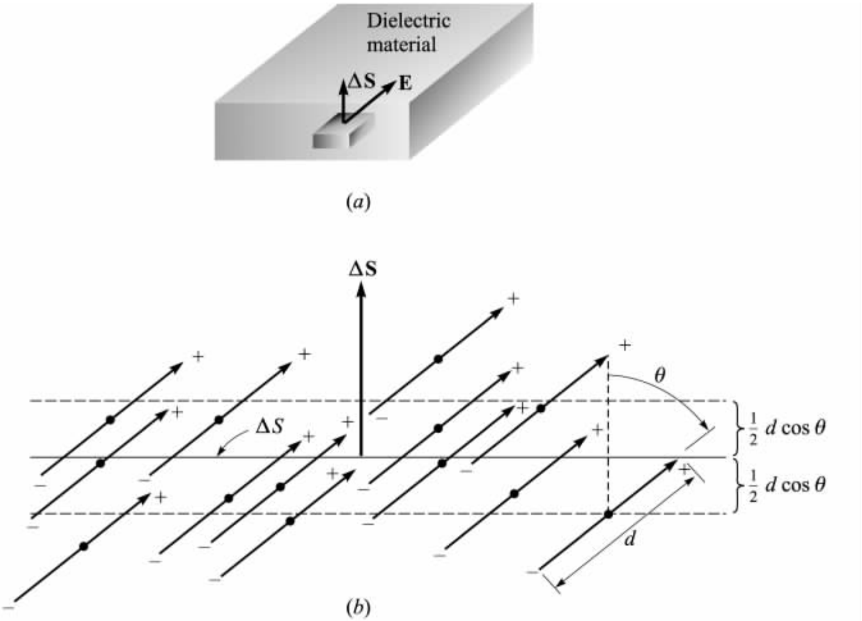

Figure 5.9 (*a*) An incremental surface element ΔS is shown in the interior of a dielectric in which an electric field $\mathbf{E}$ is present. (*b*) The nonpolar molecules form dipole moments $\mathbf{p}$ and a polarization $\mathbf{P}$. There is a net transfer of bound charge across ΔS.

$\mathbf{p} = Q\mathbf{d}$ in each molecule, such that $\mathbf{p}$ and $\mathbf{d}$ make an angle θ with $\Delta\mathbf{S}$, as indicated in Figure 5.9*b*.

The bound charges will now move across $\Delta\mathbf{S}$. Each of the charges associated with the creation of a dipole must have moved a distance $\frac{1}{2}d\cos\theta$ in the direction perpendicular to $\Delta\mathbf{S}$. Thus, any positive charges initially lying below the surface $\Delta\mathbf{S}$ and within the distance $\frac{1}{2}d\cos\theta$ of the surface must have crossed $\Delta\mathbf{S}$ going upward. Also, any negative charges initially lying above the surface and within that distance ($\frac{1}{2}d\cos\theta$) from $\Delta\mathbf{S}$ must have crossed $\Delta\mathbf{S}$ going downward. Therefore, because there are n molecules/m^3, the net total charge that crosses the elemental surface in an upward direction is equal to $nQd\cos\theta\,\Delta S$, or

$$\Delta Q_b = nQ\mathbf{d}\cdot\Delta\mathbf{S}$$

where the subscript on Q_b reminds us that we are dealing with a bound charge and not a free charge. In terms of the polarization, we have

$$\Delta Q_b = \mathbf{P}\cdot\Delta\mathbf{S}$$

If we interpret $\Delta\mathbf{S}$ as an element of a *closed* surface inside the dielectric material, then the direction of $\Delta\mathbf{S}$ is outward, and the net increase in the bound charge *within* the closed surface is obtained through the integral

$$Q_b = -\oint_S \mathbf{P}\cdot d\mathbf{S} \qquad (22)$$

This last relationship has some resemblance to Gauss's law, and we may now generalize our definition of electric flux density so that it applies to media other than free space. We first write Gauss's law in terms of $\epsilon_0\mathbf{E}$ and Q_T, the *total* enclosed charge, bound plus free:

$$Q_T = \oint_S \epsilon_0 \mathbf{E} \cdot d\mathbf{S} \tag{23}$$

where

$$Q_T = Q_b + Q$$

and Q is the total *free* charge enclosed by the surface S. Note that the free charge appears without a subscript because it is the most important type of charge and will appear in Maxwell's equations.

Combining these last three equations, we obtain an expression for the free charge enclosed,

$$Q = Q_T - Q_b = \oint_S (\epsilon_0 \mathbf{E} + \mathbf{P}) \cdot d\mathbf{S} \tag{24}$$

D is now defined in more general terms than was done in Chapter 3,

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}} \tag{25}$$

There is thus an added term to **D** that appears when polarizable material is present. Thus,

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S} \tag{26}$$

where Q is the free charge enclosed.

Utilizing the several volume charge densities, we have

$$Q_b = \int_v \rho_b \, dv$$

$$Q = \int_v \rho_v \, dv$$

$$Q_T = \int_v \rho_T \, dv$$

With the help of the divergence theorem, we may therefore transform Eqs. (22), (23), and (26) into the equivalent divergence relationships,

$$\nabla \cdot \mathbf{P} = -\rho_b$$

$$\nabla \cdot \epsilon_0 \mathbf{E} = \rho_T$$

$$\boxed{\nabla \cdot \mathbf{D} = \rho_v} \tag{27}$$

We will emphasize only Eq. (26) and (27), the two expressions involving the free charge, in the work that follows.

In order to make any real use of these new concepts, it is necessary to know the relationship between the electric field intensity **E** and the polarization **P** that results. This relationship will, of course, be a function of the type of material, and we will essentially limit our discussion to those isotropic materials for which **E** and **P** are linearly related. In an isotropic material, the vectors **E** and **P** are always parallel, regardless of the orientation of the field. Although most engineering dielectrics are linear for moderate-to-large field strengths and are also isotropic, single crystals may be anisotropic. The periodic nature of crystalline materials causes dipole moments to be formed most easily along the crystal axes, and not necessarily in the direction of the applied field.

In *ferroelectric* materials, the relationship between **P** and **E** not only is nonlinear, but also shows hysteresis effects; that is, the polarization produced by a given electric field intensity depends on the past history of the sample. Important examples of this type of dielectric are barium titanate, often used in ceramic capacitors, and Rochelle salt.

The linear relationship between **P** and **E** is

$$\boxed{\mathbf{P} = \chi_e \epsilon_0 \mathbf{E}} \tag{28}$$

where χ_e (chi) is a dimensionless quantity called the *electric susceptibility* of the material.

Using this relationship in Eq. (25), we have

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \chi_e \epsilon_0 \mathbf{E} = (\chi_e + 1)\epsilon_0 \mathbf{E}$$

The expression within the parentheses is now defined as

$$\epsilon_r = \chi_e + 1 \tag{29}$$

This is another dimensionless quantity, and it is known as the *relative permittivity*, or *dielectric constant* of the material. Thus,

$$\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E} = \epsilon \mathbf{E} \tag{30}$$

where

$$\boxed{\epsilon = \epsilon_0 \epsilon_r} \tag{31}$$

and ϵ is the *permittivity*. The dielectric constants are given for some representative materials in Appendix C.

Anisotropic dielectric materials cannot be described in terms of a simple susceptibility or permittivity parameter. Instead, we find that each component of **D** may be a function of every component of **E**, and $\mathbf{D} = \epsilon \mathbf{E}$ becomes a matrix equation where **D** and **E** are each 3 × 1 column matrices and ϵ is a 3 × 3 square matrix. Expanding the matrix equation gives

$$\begin{aligned}
D_x &= \epsilon_{xx} E_x + \epsilon_{xy} E_y + \epsilon_{xz} E_z \\
D_y &= \epsilon_{yx} E_x + \epsilon_{yy} E_y + \epsilon_{yz} E_z \\
D_z &= \epsilon_{zx} E_x + \epsilon_{zy} E_y + \epsilon_{zz} E_z
\end{aligned}$$

Note that the elements of the matrix depend on the selection of the coordinate axes in the anisotropic material. Certain choices of axis directions lead to simpler matrices.[7]

D and **E** (and **P**) are no longer parallel, and although $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ remains a valid equation for anisotropic materials, we may continue to use $\mathbf{D} = \epsilon\mathbf{E}$ only by interpreting it as a matrix equation. We will concentrate our attention on linear isotropic materials and reserve the general case for a more advanced text.

In summary, then, we now have a relationship between **D** and **E** that depends on the dielectric material present,

$$\boxed{\mathbf{D} = \epsilon\mathbf{E}} \tag{30}$$

where

$$\boxed{\epsilon = \epsilon_0\epsilon_r} \tag{31}$$

This electric flux density is still related to the free charge by either the point or integral form of Gauss's law:

$$\boxed{\nabla \cdot \mathbf{D} = \rho_v} \tag{27}$$

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{S} = Q} \tag{26}$$

Use of the relative permittivity, as indicated by Eq. (31), makes consideration of the polarization, dipole moments, and bound charge unnecessary. However, when anisotropic or nonlinear materials must be considered, the relative permittivity, in the simple scalar form that we have discussed, is no longer applicable.

EXAMPLE 5.4

We locate a slab of Teflon in the region $0 \le x \le a$, and assume free space where $x < 0$ and $x > a$. Outside the Teflon there is a uniform field $\mathbf{E}_{\text{out}} = E_0\mathbf{a}_x$ V/m. We seek values for **D**, **E**, and **P** everywhere.

Solution. The dielectric constant of the Teflon is 2.1, and thus the electric susceptibility is 1.1.

Outside the slab, we have immediately $\mathbf{D}_{\text{out}} = \epsilon_0 E_0\mathbf{a}_x$. Also, as there is no dielectric material there, $\mathbf{P}_{\text{out}} = 0$. Now, any of the last four or five equations will enable us to relate the several fields inside the material to each other. Thus

$$\mathbf{D}_{\text{in}} = 2.1\epsilon_0\mathbf{E}_{\text{in}} \qquad (0 \le x \le a)$$

$$\mathbf{P}_{\text{in}} = 1.1\epsilon_0\mathbf{E}_{\text{in}} \qquad (0 \le x \le a)$$

---

[7] A more complete discussion of this matrix may be found in the Ramo, Whinnery, and Van Duzer reference listed at the end of this chapter.

As soon as we establish a value for any of these three fields within the dielectric, the other two can be found immediately. The difficulty lies in crossing over the boundary from the known fields external to the dielectric to the unknown ones within it. To do this we need a boundary condition, and this is the subject of the next section. We will complete this example then.

In the remainder of this text we will describe polarizable materials in terms of **D** and ϵ rather than **P** and χ_e. We will limit our discussion to isotropic materials.

D5.8. A slab of dielectric material has a relative dielectric constant of 3.8 and contains a uniform electric flux density of 8 nC/m^2. If the material is lossless, find: *(a)* E; *(b)* P; *(c)* the average number of dipoles per cubic meter if the average dipole moment is $10^{-29}\text{C} \cdot \text{m}$.

Ans. 238 V/m; 5.89 nC/m^2; 5.89×10^{20} m^{-3}

5.8 BOUNDARY CONDITIONS FOR PERFECT DIELECTRIC MATERIALS

How do we attack a problem in which there are two different dielectrics, or a dielectric and a conductor? This is another example of a *boundary condition*, such as the condition at the surface of a conductor whereby the tangential fields are zero and the normal electric flux density is equal to the surface charge density on the conductor. Now we take the first step in solving a two-dielectric problem, or a dielectric-conductor problem, by determining the behavior of the fields at the dielectric interface.

Let us first consider the interface between two dielectrics having permittivities ϵ_1 and ϵ_2 and occupying regions 1 and 2, as shown in Figure 5.10. We first examine

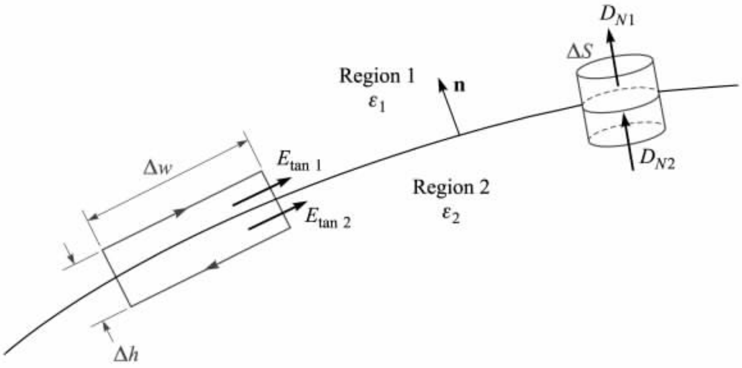

Figure 5.10 The boundary between perfect dielectrics of permittivities ϵ_1 and ϵ_2. The continuity of D_N is shown by the gaussian surface on the right, and the continuity of $E_{\tan}$ is shown by the line integral about the closed path at the left.

the tangential components by using

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

around the small closed path on the left, obtaining

$$E_{\tan 1}\,\Delta w - E_{\tan 2}\,\Delta w = 0$$

The small contribution to the line integral by the normal component of **E** along the sections of length Δh becomes negligible as Δh decreases and the closed path crowds the surface. Immediately, then,

$$\boxed{E_{\tan 1} = E_{\tan 2}} \tag{32}$$

Evidently, Kirchhoff's voltage law is still applicable to this case. Certainly we have shown that the potential difference between any two points on the boundary that are separated by a distance Δw is the same immediately above or below the boundary.

If the tangential electric field intensity is continuous across the boundary, then tangential **D** is discontinuous, for

$$\frac{D_{\tan 1}}{\epsilon_1} = E_{\tan 1} = E_{\tan 2} = \frac{D_{\tan 2}}{\epsilon_2}$$

or

$$\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{\epsilon_1}{\epsilon_2} \tag{33}$$

The boundary conditions on the normal components are found by applying Gauss's law to the small "pillbox" shown at the right in Figure 5.10. The sides are again very short, and the flux leaving the top and bottom surfaces is the difference

$$D_{N1}\Delta S - D_{N2}\Delta S = \Delta Q = \rho_S \Delta S$$

from which

$$\boxed{D_{N1} - D_{N2} = \rho_S} \tag{34}$$

What is this surface charge density? It cannot be a *bound* surface charge density, because we are taking the polarization of the dielectric into effect by using a dielectric constant different from unity; that is, instead of considering bound charges in free space, we are using an increased permittivity. Also, it is extremely unlikely that any *free* charge is on the interface, for no free charge is available in the perfect dielectrics we are considering. This charge must then have been placed there deliberately, thus unbalancing the total charge in and on this dielectric body. Except for this special case, then, we may assume ρ_S is zero on the interface and

$$\boxed{D_{N1} = D_{N2}} \tag{35}$$

or the normal component of **D** is continuous. It follows that

$$\epsilon_1 E_{N1} = \epsilon_2 E_{N2} \tag{36}$$

and normal **E** is discontinuous.

Equations (32) and (34) can be written in terms of field vectors in any direction, along with the unit normal to the surface as shown in Figure 5.10. Formally stated, the boundary conditions for the electric flux density and the electric field strength at the surface of a perfect dielectric are

$$(\mathbf{D}_1 - \mathbf{D}_2) \cdot \mathbf{n} = \rho_s \tag{37}$$

which is the general statement of Eq. (32), and

$$(\mathbf{E}_1 - \mathbf{E}_2) \times \mathbf{n} = 0 \tag{38}$$

generally states Eq. (34). This construction was used previously in Eqs. (17) and (18) for a conducting surface, in which the normal or tangential components of the fields are obtained through the dot product or cross product with the normal, respectively.

These conditions may be used to show the change in the vectors **D** and **E** at the surface. Let $\mathbf{D}_1$ (and $\mathbf{E}_1$) make an angle θ_1 with a normal to the surface (Figure 5.11). Because the normal components of **D** are continuous,

$$D_{N1} = D_1 \cos\theta_1 = D_2 \cos\theta_2 = D_{N2} \tag{39}$$

The ratio of the tangential components is given by (33) as

$$\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{D_1 \sin\theta_1}{D_2 \sin\theta_2} = \frac{\epsilon_1}{\epsilon_2}$$

or

$$\epsilon_2 D_1 \sin\theta_1 = \epsilon_1 D_2 \sin\theta_2 \tag{40}$$

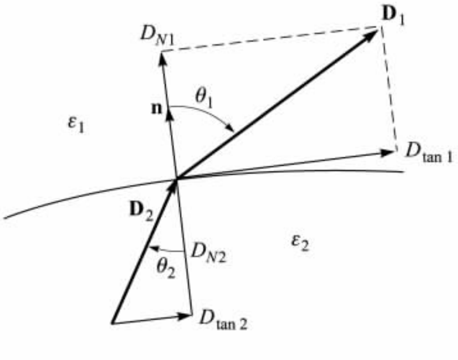

Figure 5.11 The refraction of **D** at a dielectric interface. For the case shown, $\epsilon_1 > \epsilon_2$; $\mathbf{E}_1$ and $\mathbf{E}_2$ are directed along $\mathbf{D}_1$ and $\mathbf{D}_2$, with $D_1 > D_2$ and $E_1 < E_2$.

and the division of this equation by (39) gives

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_1}{\epsilon_2} \tag{41}$$

In Figure 5.11 we have assumed that $\epsilon_1 > \epsilon_2$, and therefore $\theta_1 > \theta_2$.

The direction of **E** on each side of the boundary is identical with the direction of **D**, because $\mathbf{D} = \epsilon\mathbf{E}$.

The magnitude of **D** in region 2 may be found from Eq. (39) and (40),

$$D_2 = D_1\sqrt{\cos^2\theta_1 + \left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \sin^2\theta_1} \tag{42}$$

and the magnitude of $\mathbf{E}_2$ is

$$E_2 = E_1\sqrt{\sin^2\theta_1 + \left(\frac{\epsilon_1}{\epsilon_2}\right)^2 \cos^2\theta_1} \tag{43}$$

An inspection of these equations shows that D is larger in the region of larger permittivity (unless $\theta_1 = \theta_2 = 0^\circ$ where the magnitude is unchanged) and that E is larger in the region of smaller permittivity (unless $\theta_1 = \theta_2 = 90^\circ$, where its magnitude is unchanged).

EXAMPLE 5.5

Complete Example 5.4 by finding the fields within the Teflon ($\epsilon_r = 2.1$), given the uniform external field $\mathbf{E}_{\text{out}} = E_0\mathbf{a}_x$ in free space.

Solution. We recall that we had a slab of Teflon extending from $x = 0$ to $x = a$, as shown in Figure 5.12, with free space on both sides of it and an external field $\mathbf{E}_{\text{out}} = E_0\mathbf{a}_x$. We also have $\mathbf{D}_{\text{out}} = \epsilon_0 E_0\mathbf{a}_x$ and $\mathbf{P}_{\text{out}} = 0$.

Inside, the continuity of D_N at the boundary allows us to find that $\mathbf{D}_{\text{in}} = \mathbf{D}_{\text{out}} = \epsilon_0 E_0\mathbf{a}_x$. This gives us $\mathbf{E}_{\text{in}} = \mathbf{D}_{\text{in}}/\epsilon = \epsilon_0 E_0\mathbf{a}_x/(\epsilon_r\epsilon_0) = 0.476E_0\mathbf{a}_x$. To get the polarization field in the dielectric, we use $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ and obtain

$$\mathbf{P}_{\text{in}} = \mathbf{D}_{\text{in}} - \epsilon_0\mathbf{E}_{\text{in}} = \epsilon_0 E_0\mathbf{a}_x - 0.476\epsilon_0 E_0\mathbf{a}_x = 0.524\epsilon_0 E_0\mathbf{a}_x$$

Summarizing then gives

$$\begin{aligned}
\mathbf{D}_{\text{in}} &= \epsilon_0 E_0\mathbf{a}_x && (0 \le x \le a)\\
\mathbf{E}_{\text{in}} &= 0.476E_0\mathbf{a}_x && (0 \le x \le a)\\
\mathbf{P}_{\text{in}} &= 0.524\epsilon_0 E_0\mathbf{a}_x && (0 \le x \le a)
\end{aligned}$$

A practical problem most often does not provide us with a direct knowledge of the field on either side of the boundary. The boundary conditions must be used to help us determine the fields on both sides of the boundary from the other information that is given.

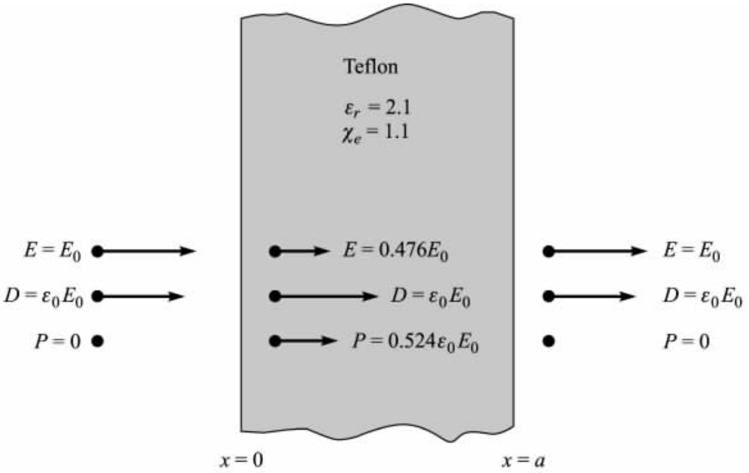

Figure 5.12 A knowledge of the electric field external to the dielectric enables us to find the remaining external fields first and then to use the continuity of normal **D** to begin finding the internal fields.

D5.9. Let Region 1 ($z < 0$) be composed of a uniform dielectric material for which $\epsilon_r = 3.2$, while Region 2 ($z > 0$) is characterized by $\epsilon_r = 2$. Let $\mathbf{D}_1 = -30\mathbf{a}_x + 50\mathbf{a}_y + 70\mathbf{a}_z$ nC/m^2 and find: (*a*) D_{N1}; (*b*) $\mathbf{D}_{t1}$; (*c*) D_{t1}; (*d*) D_1; (*e*) θ_1; (*f*) $\mathbf{P}_1$.

Ans. 70 nC/m^2; $-30\mathbf{a}_x + 50\mathbf{a}_y$ nC/m^2; 58.3 nC/m^2; 91.1 nC/m^2; 39.8°; $-20.6\mathbf{a}_x + 34.4\mathbf{a}_y + 48.1\mathbf{a}_z$ nC/m^2

D5.10. Continue Problem D5.9 by finding: (*a*) $\mathbf{D}_{N2}$; (*b*) $\mathbf{D}_{t2}$; (*c*) $\mathbf{D}_2$; (*d*) $\mathbf{P}_2$; (*e*) θ_2.

Ans. $70\mathbf{a}_z$ nC/m^2; $-18.75\mathbf{a}_x + 31.25\mathbf{a}_y$ nC/m^2; $-18.75\mathbf{a}_x + 31.25\mathbf{a}_y + 70\mathbf{a}_z$ nC/m^2; $-9.38\mathbf{a}_x + 15.63\mathbf{a}_y + 35\mathbf{a}_z$ nC/m^2; 27.5°

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CHAPTER 5 PROBLEMS

5.1 Given the current density $\mathbf{J} = -10^4[\sin(2x)e^{-2y}\mathbf{a}_x + \cos(2x)e^{-2y}\mathbf{a}_y]$ kA/m^2 (*a*) Find the total current crossing the plane $y = 1$ in the $\mathbf{a}_y$ direction in the region $0 < x < 1, 0 < z < 2$. (*b*) Find the total current leaving the region $0 < x, y < 1, 2 < z < 3$ by integrating $\mathbf{J} \cdot d\mathbf{S}$ over the surface of the cube. (*c*) Repeat part (*b*), but use the divergence theorem.

5.2 Given $\mathbf{J} = -10^{-4}(y\mathbf{a}_x + x\mathbf{a}_y)$ A/m^2, find the current crossing the $y = 0$ plane in the $-\mathbf{a}_y$ direction between $z = 0$ and 1, and $x = 0$ and 2.

5.3 Let $\mathbf{J} = 400 \sin\theta/(r^2 + 4)\, \mathbf{a}_r$ A/m^2. (*a*) Find the total current flowing through that portion of the spherical surface $r = 0.8$, bounded by $0.1\pi < \theta < 0.3\pi, 0 < \phi < 2\pi$. (*b*) Find the average value of $\mathbf{J}$ over the defined area.

5.4 If volume charge density is given as $\rho_v = (\cos \omega t)/r^2$ C/m^2 in spherical coordinates, find $\mathbf{J}$. It is reasonable to assume that $\mathbf{J}$ is not a function of θ or ϕ.

5.5 Let $\mathbf{J} = 25/\rho\mathbf{a}_\rho - 20/(\rho^2 + 0.01)\, \mathbf{a}_z$ A/m^2. (*a*) Find the total current crossing the plane $z = 0.2$ in the $\mathbf{a}_z$ direction for $\rho < 0.4$. (*b*) Calculate $\partial\rho_v/\partial t$. (*c*) Find the outward current crossing the closed surface defined by $\rho = 0.01$, $\rho = 0.4$, $z = 0$, and $z = 0.2$. (*d*) Show that the divergence theorem is satisified for $\mathbf{J}$ and the surface specified in part (*c*).

5.6 In spherical coordinates, a current density $\mathbf{J} = -k/(r \sin\theta)\, \mathbf{a}_\theta$ A/m^2 exists in a conducting medium, where k is a constant. Determine the total current in the $\mathbf{a}_z$ direction that crosses a circular disk of radius R, centered on the z axis and located at (*a*) $z = 0$; (*b*) $z = h$.

5.7 Assuming that there is no transformation of mass to energy or vice versa, it is possible to write a continuity equation for mass. (*a*) If we use the continuity equation for charge as our model, what quantities correspond to $\mathbf{J}$ and ρ_v? (*b*) Given a cube 1 cm on a side, experimental data show that the rates at which mass is leaving each of the six faces are 10.25, −9.85, 1.75, −2.00, −4.05, and 4.45 mg/s. If we assume that the cube is an incremental volume element, determine an approximate value for the time rate of change of density at its center.

5.8 A truncated cone has a height of 16 cm. The circular faces on the top and bottom have radii of 2 mm and 0.1 mm, respectively. If the material from which this solid cone is constructed has a conductivity of 2×10^6 S/m, use some good approximations to determine the resistance between the two circular faces.

5.9 (*a*) Using data tabulated in Appendix C, calculate the required diameter for a 2-m-long nichrome wire that will dissipate an average power of 450 W when 120 V rms at 60 Hz is applied to it. (*b*) Calculate the rms current density in the wire.

5.10 A large brass washer has a 2-cm inside diameter, a 5-cm outside diameter, and is 0.5 cm thick. Its conductivity is $\sigma = 1.5 \times 10^7$ S/m. The washer is cut in half along a diameter, and a voltage is applied between the two rectangular faces of one part. The resultant electric field in the interior of the half-washer is $\mathbf{E} = (0.5/\rho)\,\mathbf{a}_\phi$ V/m in cylindrical coordinates, where the z axis is the axis of the washer. (*a*) What potential difference exists between the two rectangular faces? (*b*) What total current is flowing? (*c*) What is the resistance between the two faces?

5.11 Two perfectly conducting cylindrical surfaces of length ℓ are located at $\rho = 3$ and $\rho = 5$ cm. The total current passing radially outward through the medium between the cylinders is 3 A dc. (*a*) Find the voltage and resistance between the cylinders, and **E** in the region between the cylinders, if a conducting material having $\sigma = 0.05$ S/m is present for $3 < \rho < 5$ cm. (*b*) Show that integrating the power dissipated per unit volume over the volume gives the total dissipated power.

5.12 Two identical conducting plates, each having area A, are located at $z = 0$ and $z = d$. The region between plates is filled with a material having z-dependent conductivity, $\sigma(z) = \sigma_0 e^{-z/d}$, where σ_0 is a constant. Voltage V_0 is applied to the plate at $z = d$; the plate at $z = 0$ is at zero potential. Find, in terms of the given parameters, (*a*) the resistance of the material; (*b*) the total current flowing between plates; (*c*) the electric field intensity **E** within the material.

5.13 A hollow cylindrical tube with a rectangular cross section has external dimensions of 0.5 in. by 1 in. and a wall thickness of 0.05 in. Assume that the material is brass, for which $\sigma = 1.5 \times 10^7$ S/m. A current of 200 A dc is flowing down the tube. (*a*) What voltage drop is present across a 1 m length of the tube? (*b*) Find the voltage drop if the interior of the tube is filled with a conducting material for which $\sigma = 1.5 \times 10^5$ S/m.

5.14 A rectangular conducting plate lies in the xy plane, occupying the region $0 < x < a, 0 < y < b$. An identical conducting plate is positioned directly above and parallel to the first, at $z = d$. The region between plates is filled with material having conductivity $\sigma(x) = \sigma_0 e^{-x/a}$, where σ_0 is a constant. Voltage V_0 is applied to the plate at $z = d$; the plate at $z = 0$ is at zero potential. Find, in terms of the given parameters, (*a*) the electric field intensity **E** within the material; (*b*) the total current flowing between plates; (*c*) the resistance of the material.

5.15 Let $V = 10(\rho + 1)z^2 \cos\phi$ V in free space. (*a*) Let the equipotential surface $V = 20$ V define a conductor surface. Find the equation of the conductor surface. (*b*) Find ρ and **E** at that point on the conductor surface where $\phi = 0.2\pi$ and $z = 1.5$. (*c*) Find $|\rho_S|$ at that point.

5.16 A coaxial transmission line has inner and outer conductor radii a and b. Between conductors ($a < \rho < b$) lies a conductive medium whose conductivity is $\sigma(\rho) = \sigma_0/\rho$, where σ_0 is a constant. The inner conductor is charged to potential V_0, and the outer conductor is grounded. (*a*) Assuming dc radial current I per unit length in z, determine the radial current density field **J** in A/m^2. (*b*) Determine the electric field intensity **E** in terms of I and other parameters, given or known. (*c*) By taking an appropriate line integral of **E** as found in part (*b*), find an expression that relates V_0 to I. (*d*) Find an expression for the conductance of the line per unit length, G.

5.17 Given the potential field $V = 100xz/(x^2 + 4)$ V in free space: (*a*) Find **D** at the surface $z = 0$. (*b*) Show that the $z = 0$ surface is an equipotential surface. (*c*) Assume that the $z = 0$ surface is a conductor and find the total charge on that portion of the conductor defined by $0 < x < 2$, $-3 < y < 0$.

5.18 Two parallel circular plates of radius a are located at $z = 0$ and $z = d$. The top plate ($z = d$) is raised to potential V_0; the bottom plate is grounded. Between the plates is a conducting material having radial-dependent conductivity, $\sigma(\rho) = \sigma_0\rho$, where σ_0 is a constant. (*a*) Find the ρ-*independent* electric field strength, **E**, between plates. (*b*) Find the current density, **J** between plates. (*c*) Find the total current, I, in the structure. (*d*) Find the resistance between plates.

5.19 Let $V = 20x^2yz - 10z^2$ V in free space. (*a*) Determine the equations of the equipotential surfaces on which $V = 0$ and 60 V. (*b*) Assume these are conducting surfaces and find the surface charge density at that point on the $V = 60$ V surface where $x = 2$ and $z = 1$. It is known that $0 \le V \le 60$ V is the field-containing region. (*c*) Give the unit vector at this point that is normal to the conducting surface and directed toward the $V = 0$ surface.

5.20 Two point charges of $-100\pi\,\mu$C are located at $(2, -1, 0)$ and $(2, 1, 0)$. The surface $x = 0$ is a conducting plane. (*a*) Determine the surface charge density at the origin. (*b*) Determine ρ_S at $P(0, h, 0)$.

5.21 Let the surface $y = 0$ be a perfect conductor in free space. Two uniform infinite line charges of 30 nC/m each are located at $x = 0$, $y = 1$, and $x = 0$, $y = 2$. (*a*) Let $V = 0$ at the plane $y = 0$, and find V at $P(1, 2, 0)$. (*b*) Find **E** at P.

5.22 The line segment $x = 0$, $-1 \le y \le 1$, $z = 1$, carries a linear charge density $\rho_L = \pi|y|\,\mu$C/m. Let $z = 0$ be a conducting plane and determine the surface charge density at: (*a*) $(0, 0, 0)$; (*b*) $(0, 1, 0)$.

5.23 A dipole with $\mathbf{p} = 0.1\mathbf{a}_z\ \mu\text{C}\cdot\text{m}$ is located at $A(1, 0, 0)$ in free space, and the $x = 0$ plane is perfectly conducting. (*a*) Find V at $P(2, 0, 1)$. (*b*) Find the equation of the 200 V equipotential surface in rectangular coordinates.

5.24 At a certain temperature, the electron and hole mobilities in intrinsic germanium are given as 0.43 and 0.21 $\text{m}^2/\text{V}\cdot\text{s}$, respectively. If the electron and hole concentrations are both $2.3 \times 10^{19}\ \text{m}^{-3}$, find the conductivity at this temperature.

5.25 Electron and hole concentrations increase with temperature. For pure silicon, suitable expressions are $\rho_h = -\rho_e = 6200T^{1.5}e^{-7000/T}\ \text{C/m}^3$. The functional dependence of the mobilities on temperature is given by $\mu_h = 2.3 \times 10^5 T^{-2.7}\ \text{m}^2/\text{V}\cdot\text{s}$ and $\mu_e = 2.1 \times 10^5 T^{-2.5}\ \text{m}^2/\text{V}\cdot\text{s}$, where the temperature, T, is in degrees Kelvin. Find σ at: (*a*) 0°C; (*b*) 40°C; (*c*) 80°C.

5.26 A semiconductor sample has a rectangular cross section 1.5 by 2.0 mm, and a length of 11.0 mm. The material has electron and hole densities of 1.8×10^{18} and $3.0 \times 10^{15}\ \text{m}^{-3}$, respectively. If $\mu_e = 0.082\ \text{m}^2/\text{V}\cdot\text{s}$ and $\mu_h = 0.0021\ \text{m}^2/\text{V}\cdot\text{s}$, find the resistance offered between the end faces of the sample.

5.27 Atomic hydrogen contains 5.5×10^{25} atoms/m^3 at a certain temperature and pressure. When an electric field of 4 kV/m is applied, each dipole formed by the electron and positive nucleus has an effective length of 7.1×10^{-19} m. (*a*) Find P. (*b*) Find ϵ_r.

5.28 Find the dielectric constant of a material in which the electric flux density is four times the polarization.

5.29 A coaxial conductor has radii $a = 0.8$ mm and $b = 3$ mm and a polystyrene dielectric for which $\epsilon_r = 2.56$. If $\mathbf{P} = (2/\rho)\mathbf{a}_\rho\ \text{nC/m}^2$ in the dielectric, find (*a*) $\mathbf{D}$ and $\mathbf{E}$ as functions of ρ; (*b*) V_{ab} and χ_e. (*c*) If there are 4×10^{19} molecules per cubic meter in the dielectric, find $\mathbf{p}(\rho)$.

5.30 Consider a composite material made up of two species, having number densities N_1 and N_2 molecules/m^3, respectively. The two materials are uniformly mixed, yielding a total number density of $N = N_1 + N_2$. The presence of an electric field $\mathbf{E}$ induces molecular dipole moments $\mathbf{p}_1$ and $\mathbf{p}_2$ within the individual species, whether mixed or not. Show that the dielectric constant of the composite material is given by $\epsilon_r = f\epsilon_{r1} + (1 - f)\epsilon_{r2}$, where f is the number fraction of species 1 dipoles in the composite, and where ϵ_{r1} and ϵ_{r2} are the dielectric constants that the unmixed species would have if each had number density N.

5.31 The surface $x = 0$ separates two perfect dielectrics. For $x > 0$, let $\epsilon_r = \epsilon_{r1} = 3$, while $\epsilon_{r2} = 5$ where $x < 0$. If $\mathbf{E}_1 = 80\mathbf{a}_x - 60\mathbf{a}_y - 30\mathbf{a}_z$ V/m, find (*a*) E_{N1}; (*b*) $\mathbf{E}_{T1}$; (*c*) $\mathbf{E}_1$; (*d*) the angle θ_1 between $\mathbf{E}_1$ and a normal to the surface; (*e*) D_{N2}; (*f*) D_{T2}; (*g*) $\mathbf{D}_2$; (*h*) $\mathbf{P}_2$; (*i*) the angle θ_2 between $\mathbf{E}_2$ and a normal to the surface.

5.32 Two equal but opposite-sign point charges of 3 μC are held x meters apart by a spring that provides a repulsive force given by $F_{sp} = 12(0.5 - x)$ N. Without any force of attraction, the spring would be fully extended to 0.5 m. (*a*) Determine the charge separation. (*b*) What is the dipole moment?

5.33 Two perfect dielectrics have relative permittivities $\epsilon_{r1} = 2$ and $\epsilon_{r2} = 8$. The planar interface between them is the surface $x - y + 2z = 5$. The origin lies in region 1. If $\mathbf{E}_1 = 100\mathbf{a}_x + 200\mathbf{a}_y - 50\mathbf{a}_z$ V/m, find $\mathbf{E}_2$.

5.34 Region 1 ($x \geq 0$) is a dielectric with $\epsilon_{r1} = 2$, while region 2($x < 0$) has $\epsilon_{r2} = 5$. Let $\mathbf{E}_1 = 20\mathbf{a}_x - 10\mathbf{a}_y + 50\mathbf{a}_z$ V/m. (*a*) Find $\mathbf{D}_2$. (*b*) Find the energy density in both regions.

5.35 Let the cylindrical surfaces $\rho = 4$ cm and $\rho = 9$ cm enclose two wedges of perfect dielectrics, $\epsilon_{r1} = 2$ for $0 < \phi < \pi/2$ and $\epsilon_{r2} = 5$ for $\pi/2 < \phi < 2\pi$. If $\mathbf{E}_1 = (2000/\rho)\mathbf{a}_\rho$ V/m, find (*a*) $\mathbf{E}_2$; (*b*) the total electrostatic energy stored in a 1 m length of each region.

第 6 章术语及导读

术　语

capacitance　电容，单位是法拉，简称法（F）。电容是储存电荷或电场能的元件，与电介质中的电场关联。

Poisson’s and Laplace’s equations　泊松方程和拉普拉斯方程。

导　读

1. 虽然电容的定义式是 $C=Q/V$，但电容却是故有存在的，不因是否加电压而改变（假设介质的电容率不变），因此，无论是设电荷求电压或设电压求电流，则只要找到电压与电流的关系，电容便可求得。
2. 电位所满足的泊松方程或拉普拉斯方程加上场域的边界条件称为电位的边值问题，它是用电位描述静电场的方程。通过求解边值问题可以得到电位函数，进而可求得场强。
3. 因为静电场是保守场，所以可以利用一个标量函数即电位来表示，令电位的负梯度等于电场强度，就自动保证了静电场的一个方程 $\nabla\times\mathbf{E}=0$ 得到满足，因为对任意一个标量函数均有 $\nabla\times(\nabla\varphi)=0$。因此，仅需电位再满足另外一个方程 $\nabla\cdot\mathbf{D}=\rho$，就可以完整表示静电场问题。