Chapter 1

Introduction

In this chapter we give some basic concepts of nonlinear hyperbolic system: genuinely nonlinear, linearly degenerate, weak linear degenerate, matching condition etc.

1.1 Intention and Significances

For the following nonlinear hyperbolic system:

$$\boldsymbol{u}_t + \boldsymbol{A}(\boldsymbol{u})\boldsymbol{u}_x = \boldsymbol{B}(\boldsymbol{u}) \tag{1.1.1}$$

where $\boldsymbol{u} = (u_1, u_2, \cdots, u_n)^{\mathrm{T}}$ is unknown vector function, $\boldsymbol{B}(\boldsymbol{u}) \in C^1(\mathbf{R}^n)$ is known vector function with $\boldsymbol{B}(\boldsymbol{u}) = (b_1(\boldsymbol{u}), \cdots, b_n(\boldsymbol{u}))^{\mathrm{T}}$, and $\boldsymbol{A}(\boldsymbol{u}) = (a_{ij}(\boldsymbol{u}))_{n\times n}(a_{ij}(\boldsymbol{u}) \in C^1(\mathbf{R}^n),\ i,j = 1,2,\cdots,n)$ is known matrix function, it is well-known that system (1.1.1) may be arisen in many physics, such as nonlinear wave phenomena, gas dynamics system, elastic dynamics, the kinetic theory and multiphase flow. These equations play an important role in both science (such as physics, mechanics, biology, etc.) and technology.

If the matrix $\boldsymbol{A}(\boldsymbol{u})$ is independent of $\boldsymbol{u}$, we meet linear hyperbolic waves given by

$$\boldsymbol{u}_t + \boldsymbol{A}\boldsymbol{u}_x = \boldsymbol{B}(\boldsymbol{u})$$

In the scalar case, we have, for instance, the Cauchy problem

$$\begin{cases} u_t + u_x = 0 \\ t = 0 : u = \phi(x) \end{cases}$$

where $\phi(x) \in C^1$ with bounded C^1 norm. The classical solution always exists for $t \in \mathbf{R}$, that is, the wave speed is constant: $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 1$ and the wave always keeps its shape in the course of propagation. In the general case, there are n linear waves given by

$$\begin{aligned} &\boldsymbol{u}_t + \boldsymbol{A}\boldsymbol{u}_x = \boldsymbol{0} \\ &t = 0 : \boldsymbol{u} = \boldsymbol{\phi}(x) \end{aligned}$$

with constant speeds

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_i \ (i = 1, 2, \cdots, n)$$

where λ_i is the eigenvalue of the matrix $\boldsymbol{A}$ and $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. Each wave keeps its shape in the propagation, and the interaction among waves is only a linear superposition. It is the reason that we can hear and distinguish many persons speaking at the same time. Otherwise, our life would be very complicated.

The situation for nonlinear hyperbolic system is totally different. Generally speaking, the classical solutions to system (1.1.1) exists only locally in time and singularities may occur in a finite time, even if the initial data are sufficiently smooth or sufficiently small. To illustrate this, we give two simple examples.

Example 1.1.1 Consider the following Cauchy problem of Burger's equation with inhomogeneous term:

$$\begin{aligned} &u_t + uu_x = u^2 \\ &t = 0 : u = u_0(x) \end{aligned} \tag{1.1.2}$$

where $u_0(x) \in C_0^2([a, b]), u_0(x)$ exists maximum value at the point $\beta_0 \in (a, b)$, and

$$u_0(\beta_0) > 0, \quad u_0''(\beta_0) \neq 0$$

On the existence domain $\{(t, x) | 0 \leqslant t \leqslant T_0, x \in \mathbf{R}\}$ of the classical solution to Cauchy problem (1.1.2), let

$$x = \phi(t, \beta), \quad \phi(0, \beta) = \beta$$

be characteristics, and

$$v(t, \beta) = u(t, \phi(t, \beta))$$

then, (ϕ, v) satisfies

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = v, \quad \frac{\mathrm{d}v}{\mathrm{d}t} = v^2, \quad \phi(0,\beta) = \beta, \quad v(0,\beta) = u_0(\beta) \tag{1.1.3}$$

It follows from (1.1.3) that

$$u(t,x) = v(t,\beta) = \frac{u_0(\beta)}{1 - tu_0(\beta)} \tag{1.1.4}$$

Obviously, the life span $\bar{T}$ for $v(t,\beta)$ satisfies

$$\bar{T}' \overset{\text{def}}{=} \frac{1}{\max u_0(\beta)}$$

Moreover, we have

$$\phi(t,\beta) = \beta - \ln(1 - tu_0(\beta))$$

Hence,

$$\frac{\partial \phi}{\partial \beta} = 1 + \frac{tu_0(\beta)}{1 - tu_0(\beta)} \tag{1.1.5}$$

Suppose that $\partial_x u$ blows up at $t = T^{*\prime} > 0$. Since

$$\frac{\partial u}{\partial x} \to \infty / \frac{\partial \phi}{\partial \beta} \to 0$$

as $t \to T^{*\prime}$. Thus, we obtain

$$T^{*\prime} = h(\beta) \overset{\text{def}}{=} \frac{1}{u_0(\beta) - u_0(\beta)}$$

By $u_0(\beta_0) = 0, u_0(\beta_0) = 0$, we have

$$h\,(\beta_0) = 0$$

Noting the continuity of $h(\beta)$,there exists a neighborhood domain $D(\beta_0)$ of β_0, such that

$$h\,(\beta) = 0, \quad \beta \in D(\beta_0)$$

Without loss of generality, we suppose that

$$h\,(\beta) > 0, \quad \beta \in D(\beta_0)$$

Then, there exists $\beta_{*}\in D(\beta_0)$, such that

$$h(\beta_*) < h(\beta_0)$$

that is,

$$T^{*\prime} < \bar{T}^{\prime} \tag{1.1.6}$$

(1.1.6) shows that we can choose suitably $u_0(x)$ such that $u_x(t,x)$ first blows up in a finite time.

On the other hand, by (1.1.4) and (1.1.5), if $u_0(x) \in C^1(\mathbf{R})$, and

$$u_0(x) \leqslant 0, \quad u_0(x) \geqslant 0, \quad \forall x \in \mathbf{R}$$

then Cauchy problem (1.1.2) admits a unique global classical solution on $t \geqslant 0$.

Example 1.1.2 Consider nonlinear hyperbolic system with dissipation:

$$\begin{aligned} & u_t + uu_x = -\alpha u \\ & t = 0 : u = \phi(x) \end{aligned} \tag{1.1.7}$$

where α $(\alpha > 0)$ is a constant,$\phi(x) \in C^1(\mathbf{R})$ with bounded C^1 norm.

Suppose that $x = x(t,\beta)$ $(x(0,\beta) = \beta)$ is characteristics, then, we have

$$u(t,x) = \phi(\beta)\exp(-\alpha t)$$

$$u_x(t,x) = \frac{\phi\ (\beta)\exp(-\alpha t)}{1+\alpha^{-1}\phi\ (\beta)(1-\exp(-\alpha t))} \tag{1.1.8}$$

By (1.1.8), if α $(\alpha > 0)$ is suitably large, then $\partial_x u(t,x)$ admits uniform *a priori* estimate, and then, Cauchy problem (1.1.7) admits a unique global classical solution on $t \geqslant 0$. If α $(\alpha > 0)$ is suitably small, then there exists $T_0 > 0$ (depending on β and α), such that

$$u_x(t,x) \to \infty$$

as $t \to T_0^{-\prime}$. Hence, the classical solution to Cauchy problem (1.1.7) must blow up in a finite time.

There is considerable practical interest in obtaining numerical approximations of solution to system (1.1.1). Knowing that the solution is smooth and allows one to take advantage of efficient high-order schemes which may be in appropriate for solutions with discontinuity. In fact, the global existence of the approximate finite element solution shows that the approximate solution is always in a neighborhood of a classical solution to system (1.1.1).

Therefore, for the first order nonlinear hyperbolic system (1.1.1), it is of great important in both theory and application to study the following three problems.

(1) *Under what conditions, does the problem under consideration* (*Cauchy problem, Boundary value problem, Generalized Riemann problem etc.*) *for the first order nonlinear hyperbolic system admit a unique global classical solution on* $t \geqslant 0$? *Basing on this problem, we can further study the regularity and the global behavior of the solution, particularly the asymptotic behavior of the solution as* $t \to +\infty$.

(2) *Under what conditions, does the classical solution to the problem under consideration blow up in a finite time? When and where does the solution blow up? Which quantities will blow up? Can we further investigate the behavior or mechanisms of the blow-up phenomenon?*

Even if the solution blows up in a finite time, physical phenomenon still exists with singularities. Therefore one wants to understand further.

(3) *How do the singularities, in particular, shocks grow out of nothing? What is the structure of the singularities? What about the stability of the singularities?*

For the case that $n = 1$ or $n = 2$, these problems have been solved completely by the method of characteristics and the Whitney's theory of singularities of mapping of the plane into the plane (cf. [33] and the references therein).

For the following simple and important case:

$$\boldsymbol{u}_t + \boldsymbol{A}(\boldsymbol{u})\boldsymbol{u}_x = \boldsymbol{0} \tag{1.1.9}$$

Suppose that system (1.1.9) is strictly hyperbolic and genuinely nonlinear.

Consider Cauchy problem of system (1.1.9) with the following initial data:

$$t = 0 : \boldsymbol{u} = \boldsymbol{\phi}(x) \tag{1.1.10}$$

F.John[21] proved that if $\boldsymbol{A}(\boldsymbol{u}), \boldsymbol{\phi}(x) \in C^2, \mathrm{supp}\boldsymbol{\phi}(x) \subseteq [\alpha_0, \beta_0]$, and

$$\theta = (\beta_0 - \alpha_0)^2 \sup_x |\boldsymbol{\phi}\ \ (x)| > 0$$

is small enough, then the first order derivatives of C^2 solution $\boldsymbol{u} = \boldsymbol{u}(t,x)$ to Cauchy problem (1.1.9)-(1.1.10) must blow up in a finite time. Liu Taiping[62] generalized F. John's result to the case that a part of eigenvalues is genuinely nonlinear, while the other part of eigenvalues is linearly degenerate. In this situation he showed that for a quite large class of small initial data, the first order derivatives of the classical solution still blows up in a finite time. Hörmander[7] improved F. John's result, by a self-contained and somewhat simplified exposition of the method. Moreover, by determining the time of blow-up asymptotically, he gave a sharp estimate on the life span of the solution.

Bressan[3] gave a result a result on the global existence of the classical solutions as follows: Suppose that system (1.1.9) is strictly hyperbolic and linearly degenerate in the Lax, the initial data $\boldsymbol{\phi}(x)$ have a compact support and the total variation is small enough (i.e. $\mathrm{TV}(\boldsymbol{\phi}) \ll 1$), then the Cauchy problem (1.1.9)-(1.1.10) admits a unique global classical solution $\boldsymbol{u} = \boldsymbol{u}(t,x)$ for all $t \in \mathbf{R}$.

Employing the nonlinear geometrical optics, S. Alinhac[1] reconsidered the result presented by Hörmander and gave a more precise estimate on the life span.

Here, we point out the work obtained by Li Tatsien, Zhou Yi and Kong Dexing (cf. [27],[28], [33], [35]~[37]). They introduce some new concepts—**null condition** and **weak linear degeneracy**, gave a quite complete result on the global existence and the life span of C^1 solution to Cauchy problem (1.1.9)-(1.1.10), where the eigenvalues of system (1.1.9) might be neither genuinely nonlinear nor linearly degenerate, and $\boldsymbol{\phi}(x)$ is small in the following sense: there exists a constant μ ($\mu > 0$) such that

$$\theta \equiv \sup_{x\in\mathbf{R}}\{(1+|x|)^{1+\mu}(|\boldsymbol{\phi}(x)| + |\boldsymbol{\phi}'(x)|)\} \tag{1.1.11}$$

is small.

For the case that

$$\boldsymbol{B}(\boldsymbol{u}) \not\equiv \mathbf{0}$$

if $\boldsymbol{B}(\boldsymbol{u})$ is linear vector value function, $\boldsymbol{B}(\mathbf{0}) = \mathbf{0}$, and

$$\boldsymbol{A} = -\boldsymbol{L}(\mathbf{0})\nabla\boldsymbol{B}(\mathbf{0})\boldsymbol{L}^{-1}(\mathbf{0}) \tag{1.1.12}$$

is weak row-diagonally dominant, where $\boldsymbol{L}(\boldsymbol{u}) = (l_{ij}(\boldsymbol{u}))$ is composed by the left eigenvectors, $\boldsymbol{L}^{-1}(\mathbf{0})$ is the inverse of $\boldsymbol{L}(\mathbf{0})$, $\|\boldsymbol{u}_0(x)\|_{C^1}$ is sufficiently small, then, Cauchy problem for system (1.1.1) admits a unique global classical solution on $t \geqslant 0$. If $\boldsymbol{B}(\boldsymbol{u})$ is nonlinear vector value function, $\boldsymbol{B}(\mathbf{0}) = \mathbf{0}$, and $\boldsymbol{A}$ is strictly row-diagonally dominant, $\|\boldsymbol{u}_0(x)\|_{C^1}$ is sufficiently small, then, Cauchy problem for system (1.1.1) admits a unique global classical solution on $t \geqslant 0$[33].

1.2 Basic Concepts

1.2.1 Definition of Nonlinear Hyperbolic Systems

Definition 1.2.1 *System* (1.1.1) *is called* **hyperbolic** *on the domain under consideration, if*

(1) $\boldsymbol{A}(\boldsymbol{u})$ *has n real eigenvalues* $\lambda_i(\boldsymbol{u})$ $(i = 1, 2, \cdots, n)$;

(2) $\boldsymbol{A}(\boldsymbol{u})$ *is diagonalizable, i.e., there exists a complete set of left* (*resp. right*) *eigenvectors* $\boldsymbol{l}_i(\boldsymbol{u}) = (l_{i1}(\boldsymbol{u}), \cdots, l_{in}(\boldsymbol{u}))$ (*resp.* $\boldsymbol{r}_i(\boldsymbol{u}) = (r_{1i}(\boldsymbol{u}), \cdots, r_{ni}(\boldsymbol{u}))^{\mathrm{T}}$) *corresponding to* $\lambda_i(\boldsymbol{u})$ $(i = 1, 2, \cdots, n)$:

$$\boldsymbol{l}_i(\boldsymbol{u})\boldsymbol{A}(\boldsymbol{u}) = \lambda_i(\boldsymbol{u})\boldsymbol{l}_i(\boldsymbol{u}) \quad (\textit{resp.} \quad \boldsymbol{A}(\boldsymbol{u})\boldsymbol{r}_i(\boldsymbol{u}) = \lambda_i(\boldsymbol{u})\boldsymbol{r}_i(\boldsymbol{u})) \tag{1.2.1}$$

we have

$$\det|l_{ij}(\boldsymbol{u})| \neq 0 \quad (\textit{resp.} \quad \det|r_{ij}(\boldsymbol{u})| \neq 0) \tag{1.2.2}$$

System (1.1.1) *is called* **strictly hyperbolic** *on a certain domain, if* $\boldsymbol{A}(\boldsymbol{u})$ *admits n real and distinct eigenvalues* $\lambda_i(\boldsymbol{u})(i = 1, 2, \cdots, n)$. *Without loss of generality, we suppose that*

$$\lambda_1(\boldsymbol{u}) < \lambda_2(\boldsymbol{u}) < \cdots < \lambda_n(\boldsymbol{u}) \tag{1.2.3}$$

Without loss of generality, we may suppose that

$$\boldsymbol{l}_i(\boldsymbol{u})\boldsymbol{r}_j(\boldsymbol{u}) \equiv \delta_{ij} \quad (i, j = 1, 2, \cdots, n) \tag{1.2.4}$$

and

$$\boldsymbol{r}_i^{\mathrm{T}}(\boldsymbol{u})\boldsymbol{r}_i(\boldsymbol{u}) \equiv 1 \quad (i = 1, 2, \cdots, n) \tag{1.2.5}$$

where δ_{ij} stands for the Kronecker's symbol.

For any strictly hyperbolic system, all $\lambda_i(\boldsymbol{u}), l_{ij}(\boldsymbol{u})$ and $r_{ij}(\boldsymbol{u})$ $(i, j = 1, 2, \cdots, n)$ are supposed to have the same regularity as $a_{ij}(\boldsymbol{u})$ $(i, j = 1, 2, \cdots, n)$.

However, it is not always the case for general hyperbolic hyperbolic system. For example, let $\boldsymbol{A}(u) = \begin{pmatrix} 0 & u \\ u^2 & 0 \end{pmatrix}$, the eigenvalues $\lambda_{1,2} = \pm u^{\frac{3}{2}} \in \not{C}^{\infty\prime}$ at $u = 0$, but $\boldsymbol{A}(u) \in \not{C}^{\infty\prime}$.

1.2.2 Genuine Nonlinearity, Linear Degeneracy and Weak Linear Degeneracy

Definition 1.2.2 *For any given simple eigenvalue* $\lambda_i(\boldsymbol{u})$ *is* **genuinely nonlinear** (*denoted by GNL*) *in the sense of P.D. Lax*[29], *if*

$$\nabla\lambda_i(\boldsymbol{u})\boldsymbol{r}_i(\boldsymbol{u}) = 0, \quad \forall \boldsymbol{u} \in \not{\mathbf{R}}^n \tag{1.2.6}$$

While $\lambda_i(\boldsymbol{u})$ *is* **linearly degenerate** (*denoted by LD*) *in the sense of P.D. Lax*[29], *if*

$$\nabla\lambda_i(\boldsymbol{u})\boldsymbol{r}_i(\boldsymbol{u}) \equiv \not{0}, \quad \forall \boldsymbol{u} \in \not{\mathbf{R}}^n \tag{1.2.7}$$

System (1.1.1) is GNL (resp. LD), if all eigenvalues are GNL (resp. LD). The following $2 \times \not{2}$ nonlinear hyperbolic system in diagonal form

$$\begin{aligned} r_t + \lambda(r,s)r_x &= 0 \\ s_t + \mu(r,x)s_x &= 0 \end{aligned} \tag{1.2.8}$$

is GNL system if and only if

$$\frac{\partial\lambda(r,s)}{\partial r} = 0, \quad \frac{\partial\mu(r,s)}{\partial s} = 0, \quad \forall (r,s) \in \not{\mathbf{R}}^2 \tag{1.2.9}$$

System (1.2.8) is LD system if and only if

$$\frac{\partial\lambda(r,s)}{\partial r} \equiv \not{0}, \quad \frac{\partial\mu(r,s)}{\partial s} \equiv \not{0}, \quad \forall (r,s) \in \not{\mathbf{R}}^2 \tag{1.2.10}$$

that is

$$\lambda(r,s) \equiv \not{\lambda}(s), \quad \mu(r,s) \equiv \not{\mu}(r) \tag{1.2.11}$$

The genuine nonlinearity and the linear degeneracy are only two extreme cases. In applications, some characteristics may be neither GNL nor LD. In such a case, it is necessary to introduce a new concept—the weak linear degeneracy (cf. [33]).

Definition 1.2.3 *The i-th* $(1 \leqslant i \leqslant n)$ *eigenvalue* $\lambda_i(\boldsymbol{u})$ *is* **weak linear degenerate** (*denoted by WLD*) *with respect to* $\boldsymbol{u} = \boldsymbol{u}_0$*, if, along the i-th characteristic trajectory* $\boldsymbol{u} = \boldsymbol{u}^{(i)}(s)$ *passing through* $\boldsymbol{u} = \boldsymbol{u}_0$*, defined by*

$$\begin{cases} \dfrac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}s} = \boldsymbol{r}_i(\boldsymbol{u}(s)) \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \end{cases} \tag{1.2.12}$$

we have

$$\nabla\lambda_i(\boldsymbol{u})\boldsymbol{r}_i(\boldsymbol{u}) \equiv 0 \quad (\forall |\boldsymbol{u} - \boldsymbol{u}_0| \text{ small})$$

namely,

$$\lambda_i(\boldsymbol{u}^{(i)}(s)) \equiv \lambda_i(\boldsymbol{u}_0), \quad (\forall |s| \text{ small})$$

For simplicity and without loss of generality, we may take $\boldsymbol{u}_0 = \boldsymbol{0}$.

If $\lambda_i(\boldsymbol{u})$ is WLD, then,

$$\lambda_i(\boldsymbol{u}^{(i)}(s)) \equiv \lambda_i(\boldsymbol{0})$$

If all eigenvalues are WLD, system (1.1.1) is called the WLD.

Obviously, if, in a neighborhood domain of $\boldsymbol{u} = \boldsymbol{u}_0$, the i-th eigenvalue $\lambda_i(\boldsymbol{u})$ is LD in the sense of P.D. Lax, then $\lambda_i(\boldsymbol{u})$ is WLD.

According to Definition 1.2.3, if λ_i $(i = 1, 2, \cdots, n)$ is not WLD, either there exists an integer $\alpha_i \geqslant 0$ such that

$$\frac{\mathrm{d}^k\lambda_i(\boldsymbol{u}^{(i)}(s))}{\mathrm{d}s^k}\Big|_{s=0} = 0 \ (k = 1, 2, \cdots, \alpha_i), \text{but } \frac{\mathrm{d}^{\alpha_i+1}\lambda_i(\boldsymbol{u}^{(i)}(s))}{\mathrm{d}s^{\alpha_i+1}}\Big|_{s=0} = 0 \tag{1.2.13}$$

or

$$\frac{\mathrm{d}^k\lambda_i(\boldsymbol{u}^{(i)}(s))}{\mathrm{d}s^k}\Big|_{s=0} = 0 \ (k = 1, 2, \cdots), \text{but } \lambda_i(\boldsymbol{u}^{(i)}(s)) \not\equiv \lambda_i(\boldsymbol{0}) \tag{1.2.14}$$

denoted by $\alpha_i = +\infty$, where $\boldsymbol{u} = \boldsymbol{u}^{(i)}(s)$ is defined by (1.2.12).

α_i is called the **non-WLD index** of the eigenvalue $\lambda_i(\boldsymbol{u})$. Obviously, if $\alpha_i = 0$, then in a neighbourhood of $\boldsymbol{u} = \boldsymbol{0}$, $\lambda_i(\boldsymbol{u})$ is GNL, and when α_i increases, $\lambda_i(\boldsymbol{u})$ is closer and closer to the WLD case.

If a strictly hyperbolic system (1.1.1) is not WLD, then there exists a nonempty set $J \subsetneqq \{1, 2, \cdots, n\}$ such that $\lambda_i(\boldsymbol{u})$ is not WLD if and only if $i \in J$.

1.2.3 Characteristic Forms

For any C^1 solution $\boldsymbol{u} = \boldsymbol{u}(t, x)$ to system (1.1.1)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_i(\boldsymbol{u}(t, x)) \tag{1.2.15}$$

is called the ***i*-th characteristic direction**, its integral curve is said to be the ***i*-th characteristics**.

Let

$$\frac{\mathrm{d}}{\mathrm{d}_i t} \equiv \frac{\partial}{\partial t} + \lambda_i(\boldsymbol{u})\frac{\partial}{\partial x}$$

then, along the i-th characteristic direction,

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}_i t} = \frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{u}}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = \boldsymbol{u}_t + \lambda_i(\boldsymbol{u})\boldsymbol{u}_x$$

Multiplying (1.1.1) by $\boldsymbol{l}_i(\boldsymbol{u})$ from the left side, and noting (1.2.1), system (1.1.1) equivalently reduces to the following system of **characteristic form**

$$\boldsymbol{l}_i(\boldsymbol{u})\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}_i t} = \boldsymbol{l}_i(\boldsymbol{u})(\boldsymbol{u}_t + \lambda_i(\boldsymbol{u})\boldsymbol{u}_x) = \boldsymbol{l}_i(\boldsymbol{u})\boldsymbol{B}(\boldsymbol{u}) \quad (i = 1, 2, \cdots, n) \tag{1.2.16}$$

or

$$\sum_{j=1}^{n} l_{ij}(\boldsymbol{u})\left(\frac{\partial u_j}{\partial t} + \lambda_i(\boldsymbol{u})\frac{\partial u_j}{\partial x}\right) = \sum_{j=1}^{n} l_{ij}(\boldsymbol{u})b_j(\boldsymbol{u}) \quad (i = 1, 2, \cdots, n) \tag{1.2.17}$$

in which the i-th equation only contains the directional derivatives of all the unknown functions along the i-th characteristic direction.

For the case that $n = 2$, it is well-known that at least in a local domain of $\boldsymbol{u}$ there exist integral factors $\pi_i(\boldsymbol{u}) = 0$ $(i = 1, 2)$, such that

$$\pi_i(\boldsymbol{u})\boldsymbol{l}_i(\boldsymbol{u})\mathrm{d}\boldsymbol{u} = \pi_i(\boldsymbol{u})(l_{i1}(\boldsymbol{u})\mathrm{d}u_1 + l_{i2}(\boldsymbol{u})\mathrm{d}u_2) \quad (i = 1, 2)$$

is a total differential $\mathrm{d}U_i (i = 1, 2)$. Hence, taking U_1 and U_2 as new unknown functions, (1.2.16) reduces to a system of **diagonal form**

$$\begin{aligned} \partial_t U_1 + \lambda_1 \partial_x U_1 &= f_1 \\ \partial_t U_2 + \lambda_2 \partial_x U_2 &= f_2 \end{aligned} \tag{1.2.18}$$

in which U_1, U_2 are called the **Riemann invariants.**