

CHAPTER 3 Electric Flux Density, Gauss's Law, and Divergence

术语

electric flux density 电通量密度。

displacement density 电位移密度，简称电位移，即为电通量密度，用首字母 **D** 表示，它是一个矢量，单位是库 [仑] / 平方米 (C / m^2)。

gauss' law 高斯通量定理，它从另一个角度描述了场与源的关系。

divergence 散度，对一个矢量函数求散度，结果是一个标量函数。

After drawing the fields described in the previous chapter and becoming familiar with the concept of the streamlines that show the direction of the force on a test charge at every point, it is appropriate to give these lines a physical significance and to think of them as *flux* lines. No physical particle is projected radially outward from the point charge, and there are no steel tentacles reaching out to attract or repel an unwary test charge, but as soon as the streamlines are drawn on paper there seems to be a picture showing "something" is present.

It is very helpful to invent an *electric flux* that streams away symmetrically from a point charge and is coincident with the streamlines and to visualize this flux wherever an electric field is present.

This chapter introduces and uses the concept of electric flux and electric flux density to again solve several of the problems presented in Chapter 2. The work here turns out to be much easier for problems that possess a high degree of spatial symmetry. ■

3.1 ELECTRIC FLUX DENSITY

At this stage, the second basic field quantity in our study is introduced: the *electric flux density*, or *electric displacement*, given the symbol, **D**. This field can be considered a companion field to **E**, the electric field intensity, as it is usually (but not always) parallel to **E**, and in the basic sense, they are both associated with electric charge. The two fields are related, but they have totally different meanings: We have already defined **E** in the context of finding forces on charges, but **D** is defined in a more direct way to the charge that is generating **E**. The two fields are related to each other through the properties of the medium in which they exist. This relation is a study on its own and will be taken up in detail in Chapter 5. In this section, we explain **D** in a historical context.

3.1.1 Faraday's Experiments on Electric Displacement

About 1837, the director of the Royal Society in London, Michael Faraday, became very interested in static electric fields and the effect of various insulating materials on these fields. This problem had been bothering him during the previous ten years when he was experimenting in his now-famous work on induced electromotive force, which we will discuss in Chapter 9. With that subject completed, he had a pair of concentric metallic spheres constructed, the outer one consisting of two hemispheres that could be firmly clamped together. He also prepared shells of insulating material (or dielectric material, or simply *dielectric*) that would occupy the entire volume between the concentric spheres.

His experiment, then, consisted essentially of the following steps:

1. With the equipment dismantled, the inner sphere was given a known positive charge.
2. The hemispheres were then clamped together around the charged sphere with about 2 cm of dielectric material between them.
3. The outer sphere was discharged by connecting it momentarily to ground.
4. The outer sphere was separated carefully, using tools made of insulating material in order not to disturb the induced charge on it, and the negative induced charge on each hemisphere was measured.

Faraday found that the total charge on the outer sphere was equal in *magnitude* to the original charge placed on the inner sphere and that this was true regardless of the dielectric material separating the two spheres. He concluded that there was some sort of "displacement" from the inner sphere to the outer which was independent of the medium; we now refer to this as *displacement*, *displacement flux*, or simply *electric flux*.

Faraday's experiments also showed, of course, that a larger positive charge on the inner sphere induced a correspondingly larger negative charge on the outer sphere, leading to a direct proportionality between the electric flux and the charge on the inner sphere. The constant of proportionality is dependent on the system of units involved, and we are fortunate in our use of SI units, because the constant is unity. If electric flux is denoted by Ψ (psi) and the total charge on the inner sphere by Q , then for Faraday's experiment

$$\Psi = Q$$

and the electric flux Ψ is measured in coulombs.

3.1.2 Electric Flux Density

More quantitative information can be obtained by considering an inner sphere of radius a and an outer sphere of radius b , with charges of Q and $-Q$, respectively (Figure 3.1). The paths of electric flux Ψ extending from the inner sphere to the outer sphere are indicated by the symmetrically distributed streamlines drawn radially from one sphere to the other.

At the surface of the inner sphere, Ψ coulombs of electric flux are produced by the charge $Q(= \Psi)$ coulombs distributed uniformly over a surface having an area of $4\pi a^2 \text{ m}^2$. The density of the flux at this surface is $\Psi/4\pi a^2$ or $Q/4\pi a^2 \text{ C/m}^2$, and this is an important new quantity.

导读

1. 综合第1与第2章的内容, 已知电荷分布求场的方法主要有两种: (1) 基于点电荷表达式采用积分的方法, 该方法适用于无限大均匀介质空间的所有问题, 但有时积分不易求解, 需要采用数值积分方法; (2) 利用高斯通量定理的积分形式求解, 该方法仅能求解一些具有对称结构的问题, 其关键是选择合适的高斯面。另外还有一种求解方法是第6章介绍的先求电位再求场强法, 可称为求解电位的边值问题方法。电位的边值问题是指电位所满足的微分方程加上场域的外边界的边界条件和不同介质交界面的界面条件所构成的电位函数求解问题。

2. 高斯通量定理的微分形式, 即式(15)的 point form of Gauss' law 描述了场域中各点上的电荷密度与场(电位移)在该点的散度之间的关系, 是描述静电场的基本方程之一。电场强度的旋度为零是描述静电场的另一个基本方程, 两者结合确定了静电场的特性。

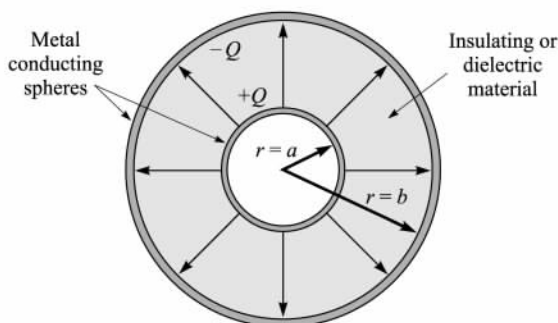


Figure 3.1 The electric flux in the region between a pair of charged concentric spheres. The direction and magnitude of \mathbf{D} are not functions of the dielectric between the spheres.

Electric flux density, measured in coulombs per square meter (sometimes described as “lines per square meter,” for each line is due to one coulomb), is given the letter \mathbf{D} , which was originally chosen because of the alternate names of *displacement flux density* or *displacement density*. Electric flux density is more descriptive, however, and we will use the term consistently.

The electric flux density \mathbf{D} is a vector field and is a member of the “flux density” class of vector fields, as opposed to the “force fields” class, which includes the electric field intensity \mathbf{E} . The direction of \mathbf{D} at a point is the direction of the flux lines at that point, and the magnitude is given by the number of flux lines crossing a surface normal to the lines divided by the surface area.

Referring again to Figure 3.1, the electric flux density is in the radial direction and has a value of

$$\begin{aligned} \mathbf{D} \Big|_{r=a} &= \frac{Q}{4\pi a^2} \mathbf{a}_r && \text{(inner sphere)} \\ \mathbf{D} \Big|_{r=b} &= \frac{Q}{4\pi b^2} \mathbf{a}_r && \text{(outer sphere)} \end{aligned}$$

and at a radial distance r , where $a \leq r \leq b$,

$$\boxed{\mathbf{D} = \frac{Q}{4\pi r^2} \mathbf{a}_r} \quad (1)$$

If we now let the inner sphere become smaller and smaller, while still retaining a charge of Q , it becomes a point charge in the limit, but the electric flux density at a point r meters from the point charge is still given by (1), for Q lines of flux are symmetrically directed outward from the point and pass through an imaginary spherical surface of area $4\pi r^2$.

This result should be compared with Section 2.2, Eq. (9), the radial electric field intensity of a point charge in free space,

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2} \mathbf{a}_r$$

In free space, therefore,

$$\mathbf{D} = \epsilon_0 \mathbf{E} \quad (\text{free space only}) \quad (2)$$

Although (2) is applicable only to a vacuum, it is not restricted solely to the field of a point charge. For a general volume charge distribution in free space, the discussion in Section 2.3.2 resulted in

$$\mathbf{E} = \int_{\text{vol}} \frac{\rho_v dv}{4\pi\epsilon_0 R^2} \mathbf{a}_R \quad (\text{free space only}) \quad (3)$$

This relationship was developed from the field of a single point charge. In a similar manner, (1) leads to

$$\mathbf{D} = \int_{\text{vol}} \frac{\rho_v dv}{4\pi R^2} \mathbf{a}_R \quad (4)$$

and (2) is therefore true for any free-space charge configuration; we will consider (2) as defining \mathbf{D} in free space.

As a preparation for the study of dielectrics later, it might be well to point out now that, for a point charge embedded in an infinite ideal dielectric medium, Faraday's results show that (1) is still applicable, and thus so is (4). Equation (3) is not applicable, however, and so the relationship between \mathbf{D} and \mathbf{E} will be slightly more complicated than (2).

Because \mathbf{D} is directly proportional to \mathbf{E} in free space, it does not seem that it should really be necessary to introduce a new symbol. We do so for a few reasons. First, \mathbf{D} is associated with the flux concept, which is an important new idea. Second, the \mathbf{D} fields we obtain will be a little simpler than the corresponding \mathbf{E} fields because ϵ_0 does not appear.

D3.1. Given a 60- μC point charge located at the origin, find the total electric flux passing through: (a) that portion of the sphere $r = 26$ cm bounded by $0 < \theta < \frac{\pi}{2}$ and $0 < \phi < \frac{\pi}{2}$; (b) the closed surface defined by $\rho = 26$ cm and $z = \pm 26$ cm; (c) the plane $z = 26$ cm.

Ans. (a) 7.5 μC ; (b) 60 μC ; (c) 30 μC

D3.2. Calculate \mathbf{D} in rectangular coordinates at point $P(2, -3, 6)$ produced by: (a) a point charge $Q_A = 55 \text{ mC}$ at $Q(-2, 3, -6)$; (b) a uniform line charge $\rho_{LB} = 20 \text{ mC/m}$ on the x axis; (c) a uniform surface charge density $\rho_{SC} = 120 \text{ } \mu\text{C/m}^2$ on the plane $z = -5 \text{ m}$.

Ans. (a) $6.38\mathbf{a}_x - 9.57\mathbf{a}_y + 19.14\mathbf{a}_z \text{ } \mu\text{C/m}^2$; (b) $-212\mathbf{a}_y + 424\mathbf{a}_z \text{ } \mu\text{C/m}^2$; (c) $60\mathbf{a}_z \text{ } \mu\text{C/m}^2$

3.2 GAUSS'S LAW

The results of Faraday's experiments with the concentric spheres could be summed up as an experimental law by stating that the electric flux passing through any imaginary spherical surface lying between the two conducting spheres is equal to the charge enclosed within that imaginary surface. This enclosed charge is distributed on the surface of the inner sphere, or it might be concentrated as a point charge at the center of the imaginary sphere. However, because one coulomb of electric flux is produced by one coulomb of charge, the inner conductor might just as well have been a cube or a brass door key and the total induced charge on the outer sphere would still be the same. Certainly the flux density would change from its previous symmetrical distribution to some unknown configuration, but $+Q$ coulombs on any inner conductor would produce an induced charge of $-Q$ coulombs on the surrounding sphere. Going one step further, we could now replace the two outer hemispheres with an empty (but completely closed) soup can. Q coulombs on the brass door key would produce $\Psi = Q$ lines of electric flux and would induce $-Q$ coulombs on the soup can.¹

These generalizations of Faraday's experiment lead to the following statement, which is known as *Gauss's law*:

The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

The contribution of Gauss, one of the greatest mathematicians the world has ever produced, was actually not in stating the law as we have, but in providing a mathematical form for this statement, which we will now obtain.

Imagine a distribution of charge, shown as a cloud of point charges in Figure 3.2, surrounded by a closed surface of any shape. The closed surface may be the surface of some real material, but more generally it is any closed surface we wish to visualize. If the total charge is Q , then Q coulombs of electric flux will pass through the enclosing surface. At every point on the surface the electric-flux-density vector \mathbf{D} will have some value \mathbf{D}_S , where the subscript S merely reminds us that \mathbf{D} must be evaluated *at the surface*, and \mathbf{D}_S will in general vary in magnitude and direction from one point on the surface to another.

Now, consider the nature of an incremental element of the surface. An incremental element of area ΔS is very nearly a portion of a plane surface, and the complete description of this surface element requires not only a statement of its magnitude ΔS but also of its orientation in space. In other words, the incremental surface element is a

¹ If it were a perfect insulator, the soup could even be left in the can without any difference in the results.

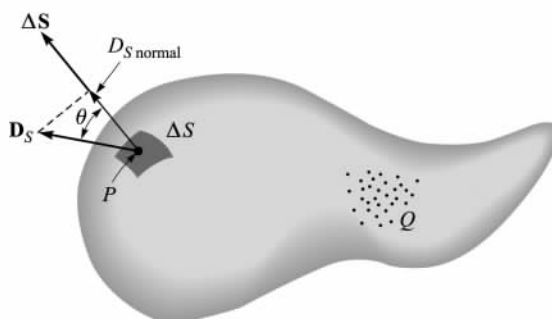


Figure 3.2 The electric flux density \mathbf{D}_S at P arising from charge Q . The total flux passing through ΔS is $\mathbf{D}_S \cdot \Delta \mathbf{S}$.

vector quantity. The only unique direction that may be associated with $\Delta \mathbf{S}$ is the direction of the normal to that plane which is tangent to the surface at the point in question. There are, of course, two such normals, and the ambiguity is removed by specifying the *outward normal* whenever the surface is closed; “outward” has a specific meaning.

At any point P , consider an incremental element of surface ΔS and let \mathbf{D}_S make an angle θ with $\Delta \mathbf{S}$, as shown in Figure 3.2. The flux crossing ΔS is then the product of the normal component of \mathbf{D}_S and ΔS ,

$$\Delta \Psi = \text{flux crossing } \Delta S = D_{S,\text{norm}} \Delta S = D_S \cos \theta \Delta S = \mathbf{D}_S \cdot \Delta \mathbf{S}$$

where we are able to apply the definition of the dot product developed in Chapter 1.

The *total* flux passing through the closed surface is obtained by adding the differential contributions crossing each surface element ΔS ,

$$\Psi = \int d\Psi = \oint_{\text{closed surface}} \mathbf{D}_S \cdot d\mathbf{S}$$

The resultant integral is a *closed surface integral*, and since the surface element $d\mathbf{S}$ always involves the differentials of two coordinates, such as $dx dy$, $\rho d\phi d\rho$, or $r^2 \sin \theta d\theta d\phi$, the integral is a double integral. Usually only one integral sign is used for brevity, and we will always place an S below the integral sign to indicate a surface integral, although this is not actually necessary, as the differential $d\mathbf{S}$ is automatically the signal for a surface integral. One last convention is to place a small circle on the integral sign itself to indicate that the integration is to be performed over a *closed* surface. Such a surface is often called a *gaussian surface*. We then have the mathematical formulation of Gauss's law,

$$\Psi = \oint_S \mathbf{D}_S \cdot d\mathbf{S} = \text{charge enclosed} = Q \quad (5)$$



The charge enclosed might be several point charges, in which case

$$Q = \sum Q_n$$

or a line charge,

$$Q = \int \rho_L dL$$

or a surface charge,

$$Q = \int_S \rho_S dS \quad (\text{not necessarily a closed surface})$$

or a volume charge distribution,

$$Q = \int_{\text{vol}} \rho_V dv$$

The last form is usually used, and we should agree now that it represents any or all of the other forms. With this understanding, Gauss's law may be written in terms of the charge distribution as

$$\oint_S \mathbf{D}_S \cdot d\mathbf{S} = \int_{\text{vol}} \rho_V dv \quad (6)$$

a mathematical statement meaning simply that the total electric flux through any closed surface is equal to the charge enclosed.

EXAMPLE 3.1

To illustrate the application of Gauss's law, let us check the results of Faraday's experiment by placing a point charge Q at the origin of a spherical coordinate system (Figure 3.3) and by choosing our closed surface as a sphere of radius a .

Solution. We have, as before,

$$\mathbf{D} = \frac{Q}{4\pi r^2} \mathbf{a}_r$$

At the surface of the sphere,

$$\mathbf{D}_S = \frac{Q}{4\pi a^2} \mathbf{a}_r$$

The differential element of area on a spherical surface is, in spherical coordinates from Chapter 1,

$$dS = r^2 \sin\theta \, d\theta \, d\phi = a^2 \sin\theta \, d\theta \, d\phi$$

or

$$d\mathbf{S} = a^2 \sin\theta \, d\theta \, d\phi \, \mathbf{a}_r$$

The integrand is

$$\mathbf{D}_S \cdot d\mathbf{S} = \frac{Q}{4\pi a^2} a^2 \sin\theta \, d\theta \, d\phi \, \mathbf{a}_r \cdot \mathbf{a}_r = \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

leading to the closed surface integral

$$\int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

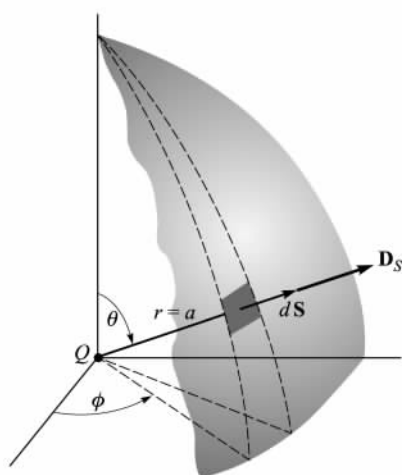


Figure 3.3 Applying Gauss's law to the field of a point charge Q on a spherical closed surface of radius a . The electric flux density \mathbf{D} is everywhere normal to the spherical surface and has a constant magnitude at every point on it.

where the limits on the integrals have been chosen so that the integration is carried over the entire surface of the sphere once.² Integrating gives

$$\int_0^{2\pi} \frac{Q}{4\pi} (-\cos\theta)_0^\pi d\phi = \int_0^{2\pi} \frac{Q}{2\pi} d\phi = Q$$

and we obtain a result showing that Q coulombs of electric flux are crossing the surface, as we should since the enclosed charge is Q coulombs.



D3.3. Given the electric flux density, $\mathbf{D} = 0.3r^2\mathbf{a}_r$ nC/m² in free space: (a) find \mathbf{E} at point $P(r = 2, \theta = 25^\circ, \phi = 90^\circ)$; (b) find the total charge within the sphere $r = 3$; (c) find the total electric flux leaving the sphere $r = 4$.

Ans. (a) $135.5\mathbf{a}_r$ V/m; (b) 305 nC; (c) 965 nC

D3.4. Calculate the total electric flux leaving the cubical surface formed by the six planes $x, y, z = \pm 5$ if the charge distribution is: (a) two point charges, $0.1 \mu\text{C}$ at $(1, -2, 3)$ and $\frac{1}{7} \mu\text{C}$ at $(-1, 2, -2)$; (b) a uniform line charge of $\pi \mu\text{C}/\text{m}$ at $x = -2, y = 3$; (c) a uniform surface charge of $0.1 \mu\text{C}/\text{m}^2$ on the plane $y = 3x$.

Ans. (a) $0.243 \mu\text{C}$; (b) $31.4 \mu\text{C}$; (c) $10.54 \mu\text{C}$

² Note that if θ and ϕ both cover the range from 0 to 2π , the spherical surface is covered twice.

3.3 APPLICATION OF GAUSS'S LAW: SOME SYMMETRICAL CHARGE DISTRIBUTIONS

We now consider how we may use Gauss's law,

$$Q = \oint_S \mathbf{D}_S \cdot d\mathbf{S}$$

to determine \mathbf{D}_S if the charge distribution is known. This is an example of an integral equation in which the unknown quantity to be determined appears inside the integral.

The solution is easy if we can choose a closed surface which satisfies two conditions:

1. \mathbf{D}_S is everywhere either normal or tangential to the closed surface, so that $\mathbf{D}_S \cdot d\mathbf{S}$ becomes either $D_S dS$ or zero, respectively.
2. On that portion of the closed surface for which $\mathbf{D}_S \cdot d\mathbf{S}$ is not zero, $D_S =$ constant.

This allows the dot product to be replaced with the product of the scalars D_S and dS , and then D_S can be brought outside the integral sign. The remaining integral is then $\int_S dS$ over that portion of the closed surface that \mathbf{D}_S crosses normally, and this is simply the area of this section of that surface. Only a knowledge of the symmetry of the problem enables us to choose such a closed surface.

3.3.1 Point Charge Field

Consider a point charge Q at the origin of a spherical coordinate system and decide on a suitable closed surface that will meet the two requirements previously listed. The surface in question is obviously a spherical surface, centered at the origin *and of any radius* r . \mathbf{D}_S is everywhere normal to the surface, and D_S has the same value at all points on the surface.

Then we have, in order,

$$\begin{aligned} Q &= \oint_S \mathbf{D}_S \cdot d\mathbf{S} = \oint_{\text{sph}} D_S dS \\ &= D_S \oint_{\text{sph}} dS = D_S \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} r^2 \sin\theta d\theta d\phi \\ &= 4\pi r^2 D_S \end{aligned}$$

and therefore

$$D_S = \frac{Q}{4\pi r^2}$$

Because r may have any value and because \mathbf{D}_S is directed radially outward,

$$\mathbf{D} = \frac{Q}{4\pi r^2} \mathbf{a}_r \quad \mathbf{E} = \frac{Q}{4\pi \epsilon_0 r^2} \mathbf{a}_r$$

which agrees with the results of Chapter 2. The example is a trivial one, and the objection could be raised that we had to know that the field was symmetrical and directed radially outward before we could obtain an answer. This is true, and that leaves the inverse-square-law relationship as the only check obtained from Gauss's law. The example does, however, serve to illustrate a method which can be applied to other problems, including several to which Coulomb's law is almost incapable of supplying an answer.

3.3.2 Line Charge Field

As a second example, consider again the uniform line charge distribution ρ_L lying along the z axis and extending from $-\infty$ to $+\infty$. We must first know the symmetry of the field, and this knowledge is complete when the answers to these two questions are known:

1. With which coordinates does the field vary (or of what variables is D a function)?
2. Which components of \mathbf{D} are present?

In using Gauss's law, it is not a question of using symmetry to simplify the solution, for the application of Gauss's law depends on symmetry, and *if we cannot show that symmetry exists then we cannot use Gauss's law* to obtain a solution. The preceding two questions now become "musts."

From our previous discussion of the uniform line charge, it is evident that only the radial component of \mathbf{D} is present, or

$$\mathbf{D} = D_\rho \mathbf{a}_\rho$$

and this component is a function of ρ only.

$$D_\rho = f(\rho)$$

The choice of a closed surface is now simple, for a cylindrical surface is the only surface to which D_ρ is everywhere normal, and it may be closed by plane surfaces normal to the z axis. A closed right circular cylinder of radius ρ extending from $z = 0$ to $z = L$ is shown in Figure 3.4.

We apply Gauss's law,

$$\begin{aligned} Q &= \oint_{\text{cyl}} \mathbf{D}_S \cdot d\mathbf{S} = D_S \int_{\text{sides}} dS + 0 \int_{\text{top}} dS + 0 \int_{\text{bottom}} dS \\ &= D_S \int_{z=0}^L \int_{\phi=0}^{2\pi} \rho \, d\phi \, dz = D_S 2\pi\rho L \end{aligned}$$

and obtain

$$D_S = D_\rho = \frac{Q}{2\pi\rho L}$$

In terms of the charge density ρ_L , the total charge enclosed is

$$Q = \rho_L L$$

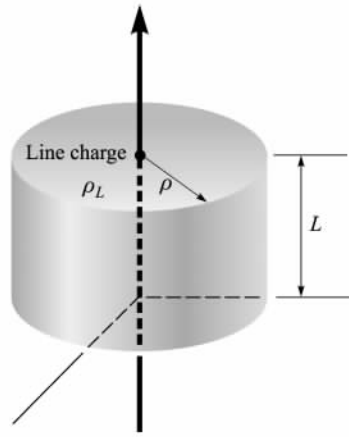


Figure 3.4 The gaussian surface for an infinite uniform line charge is a right circular cylinder of length L and radius ρ . \mathbf{D} is constant in magnitude and everywhere perpendicular to the cylindrical surface; \mathbf{D} is parallel to the end faces.

giving

$$D_\rho = \frac{\rho_L}{2\pi\rho}$$

or

$$E_\rho = \frac{\rho_L}{2\pi\epsilon_0\rho}$$

Comparing with Section 2.4, Eq. (16), shows that the correct result has been obtained and with much less work. Once the appropriate surface has been chosen, the integration usually amounts only to writing down the area of the surface at which \mathbf{D} is normal.

3.3.3 Coaxial Cable Field

The problem of a coaxial cable is almost identical to that of the line charge and is an example that is extremely difficult to solve from the standpoint of Coulomb's law. Suppose that we have two coaxial cylindrical conductors, the inner of radius a and the outer of radius b , each infinite in extent (Figure 3.5). We will assume a charge distribution of ρ_s on the outer surface of the inner conductor.

Symmetry considerations show us that only the D_ρ component is present and that it can be a function only of ρ . A right circular cylinder of length L and radius ρ , where $a < \rho < b$, is necessarily chosen as the gaussian surface, and we quickly have

$$Q = D_s 2\pi\rho L$$

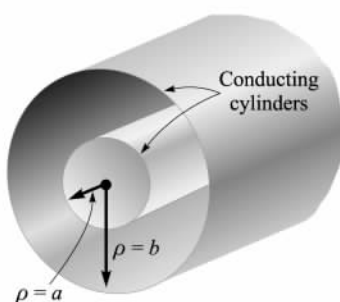


Figure 3.5 The two coaxial cylindrical conductors forming a coaxial cable provide an electric flux density within the cylinders, given by $D_\rho = a\rho_S/\rho$.

The total charge on a length L of the inner conductor is

$$Q = \int_{z=0}^L \int_{\phi=0}^{2\pi} \rho_S a \, d\phi \, dz = 2\pi a L \rho_S$$

from which we have

$$D_S = \frac{a\rho_S}{\rho} \quad \mathbf{D} = \frac{a\rho_S}{\rho} \mathbf{a}_\rho \quad (a < \rho < b)$$

This result might be expressed in terms of charge per unit length because the inner conductor has $2\pi a\rho_S$ coulombs on a meter length, and hence, letting $\rho_L = 2\pi a\rho_S$,

$$\mathbf{D} = \frac{\rho_L}{2\pi\rho} \mathbf{a}_\rho$$



and the solution has a form identical with that of the infinite line charge.

Because every line of electric flux starting from the charge on the inner cylinder must terminate on a negative charge on the inner surface of the outer cylinder, the total charge on that surface must be

$$Q_{\text{outer cyl}} = -2\pi a L \rho_{S,\text{inner cyl}}$$

and the surface charge on the outer cylinder is found as

$$2\pi b L \rho_{S,\text{outer cyl}} = -2\pi a L \rho_{S,\text{inner cyl}}$$

or

$$\rho_{S,\text{outer cyl}} = -\frac{a}{b} \rho_{S,\text{inner cyl}}$$

What would happen if we should use a cylinder of radius ρ , $\rho > b$, for the gaussian surface? The total charge enclosed would then be zero, for there are equal and opposite charges on each conducting cylinder. Hence

$$\begin{aligned} 0 &= D_S 2\pi\rho L & (\rho > b) \\ D_S &= 0 & (\rho > b) \end{aligned}$$

An identical result would be obtained for $\rho < a$. Thus the coaxial cable or capacitor has no external field (we have proved that the outer conductor is a “shield”), and there is no field within the center conductor.

Our result is also useful for a *finite* length of coaxial cable, open at both ends, provided the length L is many times greater than the radius b so that the nonsymmetrical conditions at the two ends do not appreciably affect the solution. Such a device is also termed a *coaxial capacitor*. Both the coaxial cable and the coaxial capacitor will appear frequently in the work that follows.

EXAMPLE 3.2

Let us select a 50-cm length of coaxial cable having an inner radius of 1 mm and an outer radius of 4 mm. The space between conductors is assumed to be filled with air. The total charge on the inner conductor is 30 nC. We wish to know the charge density on each conductor, and the **E** and **D** fields.

Solution. We begin by finding the surface charge density on the inner cylinder,

$$\rho_{S,\text{innercyl}} = \frac{Q_{\text{innercyl}}}{2\pi aL} = \frac{30 \times 10^{-9}}{2\pi(10^{-3})(0.5)} = 9.55 \mu\text{C/m}^2$$

The negative charge density on the inner surface of the outer cylinder is

$$\rho_{S,\text{outercyl}} = \frac{Q_{\text{outercyl}}}{2\pi bL} = \frac{-30 \times 10^{-9}}{2\pi(4 \times 10^{-3})(0.5)} = -2.39 \mu\text{C/m}^2$$

The internal fields may therefore be calculated easily:

$$D_\rho = \frac{a\rho_S}{\rho} = \frac{10^{-3}(9.55 \times 10^{-6})}{\rho} = \frac{9.55}{\rho} \text{ nC/m}^2$$

and

$$E_\rho = \frac{D_\rho}{\epsilon_0} = \frac{9.55 \times 10^{-9}}{8.854 \times 10^{-12} \rho} = \frac{1079}{\rho} \text{ V/m}$$

Both of these expressions apply to the region where $1 < \rho < 4$ mm. For $\rho < 1$ mm or $\rho > 4$ mm, **E** and **D** are zero.

D3.5. A point charge of 0.25 μC is located at $r = 0$, and uniform surface charge densities are located as follows: 2 mC/m^2 at $r = 1$ cm, and -0.6 mC/m^2 at $r = 1.8$ cm. Calculate **D** at: (a) $r = 0.5$ cm; (b) $r = 1.5$ cm; (c) $r = 2.5$ cm. (d) What uniform surface charge density should be established at $r = 3$ cm to cause **D** = 0 at $r = 3.5$ cm?

Ans. (a) $796\mathbf{a}_r \mu\text{C/m}^2$; (b) $977\mathbf{a}_r \mu\text{C/m}^2$; (c) $40.8\mathbf{a}_r \mu\text{C/m}^2$; (d) $-28.3 \mu\text{C/m}^2$

3.4 GAUSS'S LAW IN DIFFERENTIAL FORM: DIVERGENCE

We will now apply the methods of Gauss's law to a slightly different type of problem—one that may not possess any symmetry at all. At first glance, it might seem that our case is hopeless, for without symmetry, a simple gaussian surface cannot be chosen such that the normal component of \mathbf{D} is constant or zero everywhere on the surface. Without such a surface, the integral cannot be evaluated. There is only one way to circumvent these difficulties and that is to choose such a very small closed surface that \mathbf{D} is *almost* constant over the surface, and the small change in \mathbf{D} may be adequately represented by using the first two terms of the Taylor's-series expansion for \mathbf{D} . The result will become more nearly correct as the volume enclosed by the gaussian surface decreases, and we intend eventually to allow this volume to approach zero.

This example also differs from the preceding ones in that we will not obtain the value of \mathbf{D} as our answer but will instead receive some extremely valuable information about the way \mathbf{D} varies in the region of our small surface. This leads directly to one of Maxwell's four equations, which are basic to all electromagnetic theory.

3.4.1 Gauss's Law Applied to a Differential Volume Element

Consider any point P , shown in Figure 3.6, located by a rectangular coordinate system. The value of \mathbf{D} at the point P may be expressed in rectangular components, $\mathbf{D}_0 = D_{x0}\mathbf{a}_x + D_{y0}\mathbf{a}_y + D_{z0}\mathbf{a}_z$. We choose as our closed surface the small rectangular box, centered at P , having sides of lengths Δx , Δy , and Δz , and apply Gauss's law,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

In order to evaluate the integral over the closed surface, the integral must be broken up into six integrals, one over each face,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$

Consider the first of these in detail. Because the surface element is very small, \mathbf{D} is essentially constant (over *this* portion of the entire closed surface) and

$$\begin{aligned} \int_{\text{front}} &= \mathbf{D}_{\text{front}} \cdot \Delta \mathbf{S}_{\text{front}} \\ &= \mathbf{D}_{\text{front}} \cdot \Delta y \Delta z \mathbf{a}_x \\ &= D_{x,\text{front}} \Delta y \Delta z \end{aligned}$$

where we have only to approximate the value of D_x at this front face. The front face is at a distance of $\Delta x/2$ from P , and hence

$$\begin{aligned} D_{x,\text{front}} &\doteq D_{x0} + \frac{\Delta x}{2} \times \text{rate of change of } D_x \text{ with } x \\ &= D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \end{aligned}$$

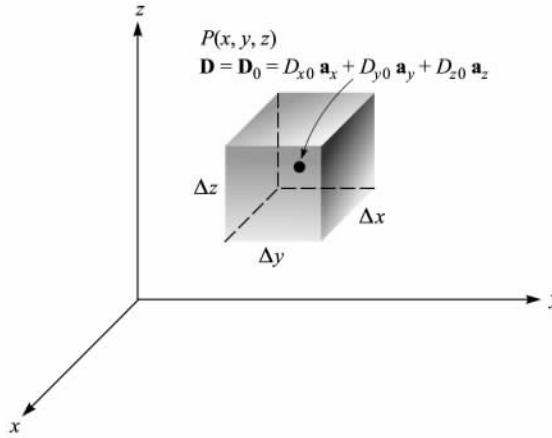


Figure 3.6 A differential-sized gaussian surface about the point P is used to investigate the space rate of change of \mathbf{D} in the neighborhood of P .

where D_{x0} is the value of D_x at P , and where a partial derivative must be used to express the rate of change of D_x with x , as D_x in general also varies with y and z . This expression could have been obtained more formally by using the constant term and the term involving the first derivative in the Taylor's-series expansion for D_x in the neighborhood of P .

We now have

$$\int_{\text{front}} \doteq \left(D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right) \Delta y \Delta z$$

Consider now the integral over the back surface,

$$\begin{aligned} \int_{\text{back}} &\doteq \mathbf{D}_{\text{back}} \cdot \Delta \mathbf{S}_{\text{back}} \\ &\doteq \mathbf{D}_{\text{back}} \cdot (-\Delta y \Delta z \mathbf{a}_x) \\ &\doteq -D_{x,\text{back}} \Delta y \Delta z \end{aligned}$$

and

$$D_{x,\text{back}} \doteq D_{x0} - \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}$$

giving

$$\int_{\text{back}} \doteq \left(-D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right) \Delta y \Delta z$$

If we combine these two integrals, we have

$$\int_{\text{front}} + \int_{\text{back}} = \frac{\partial D_x}{\partial x} \Delta x \Delta y \Delta z$$

By exactly the same process we find that

$$\int_{\text{right}} + \int_{\text{left}} = \frac{\partial D_y}{\partial y} \Delta x \Delta y \Delta z$$

and

$$\int_{\text{top}} + \int_{\text{bottom}} = \frac{\partial D_z}{\partial z} \Delta x \Delta y \Delta z$$

and these results may be collected to yield

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v \quad (7)$$

where $\Delta v = \Delta x \Delta y \Delta z$. The expression is an approximation which becomes better as Δv becomes smaller, and in the following section we shall let the volume Δv approach zero. For the moment, we have applied Gauss's law to the closed surface surrounding the volume element Δv and have as a result the approximation (7) stating that

$$\text{Charge enclosed in volume } \Delta v = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \times \text{volume } \Delta v \quad (8)$$

EXAMPLE 3.3

Find an approximate value for the total charge enclosed in an incremental volume of 10^{-9} m^3 located at the origin, if $\mathbf{D} = e^{-x} \sin y \mathbf{a}_x - e^{-x} \cos y \mathbf{a}_y + 2z \mathbf{a}_z \text{ C/m}^2$.

Solution. We first evaluate the three partial derivatives in (8):

$$\begin{aligned} \frac{\partial D_x}{\partial x} &= -e^{-x} \sin y \\ \frac{\partial D_y}{\partial y} &= e^{-x} \sin y \\ \frac{\partial D_z}{\partial z} &= 2 \end{aligned}$$

At the origin, the first two expressions are zero, and the last is 2. Thus, we find that the charge enclosed in a small volume element there must be approximately $2\Delta v$. If Δv is 10^{-9} m^3 , then we have enclosed about 2 nC.

D3.6. In free space, let $\mathbf{D} = 8xyz^4\mathbf{a}_x + 4x^2z^4\mathbf{a}_y + 16x^2yz^3\mathbf{a}_z$ pC/m². (a) Find the total electric flux passing through the rectangular surface $z = 2$, $0 < x < 2$, $1 < y < 3$, in the \mathbf{a}_z direction. (b) Find \mathbf{E} at $P(2, -1, 3)$. (c) Find an approximate value for the total charge contained in an incremental sphere located at $P(2, -1, 3)$ and having a volume of 10^{-12} m³.

Ans. (a) 1365 pC; (b) $-146.4\mathbf{a}_x + 146.4\mathbf{a}_y - 195.2\mathbf{a}_z$ V/m; (c) -2.38×10^{-21} C

3.4.2 Divergence

We next obtain an exact relationship from (7), by allowing the volume element Δv to shrink to zero. We write this equation as

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \lim_{\Delta v \rightarrow 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v} = \lim_{\Delta v \rightarrow 0} \frac{Q}{\Delta v} = \rho_v \quad (9)$$

in which the charge density, ρ_v , is identified in the second equality.

The methods of the previous section could have been used on any vector \mathbf{A} to find $\oint_S \mathbf{A} \cdot d\mathbf{S}$ for a small closed surface, leading to

$$\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) = \lim_{\Delta v \rightarrow 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} \quad (10)$$

where \mathbf{A} could represent velocity, temperature gradient, force, or any other vector field.

The operation in Eq. (10) appeared so many times in physical investigations in the nineteenth century that it received a descriptive name, *divergence*. The divergence of \mathbf{A} is defined as



$$\text{Divergence of } \mathbf{A} = \text{div } \mathbf{A} = \lim_{\Delta v \rightarrow 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} \quad (11)$$

and is usually abbreviated $\text{div } \mathbf{A}$. The physical interpretation of the divergence of a vector is obtained by describing carefully the operations implied by the right-hand side of (11), where we shall consider \mathbf{A} to be a member of the flux-density family of vectors in order to aid the physical interpretation.

The divergence of the vector flux density \mathbf{A} is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.

The physical interpretation of divergence afforded by this statement is often useful in obtaining qualitative information about the divergence of a vector field without resorting to a mathematical investigation. For instance, let us consider the divergence of the velocity of water in a bathtub after the drain has been opened. The net outflow of water through *any* closed surface lying entirely within the water must be zero, for water

is essentially incompressible, and the water entering and leaving different regions of the closed surface must be equal. Hence the divergence of this velocity is zero.

If, however, we consider the velocity of the air in a tire that has just been punctured by a nail, we realize that the air is expanding as the pressure drops, and that consequently there is a net outflow from any closed surface lying within the tire. The divergence of this velocity is therefore greater than zero.

A positive divergence for any vector quantity indicates a *source* of that vector quantity at that point. Similarly, a negative divergence indicates a *sink*. Because the divergence of the water velocity above is zero, no source or sink exists.³ The expanding air, however, produces a positive divergence of the velocity, and each interior point may be considered a source.

Writing (9) with our new term, we have

$$\text{div } \mathbf{D} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \quad (\text{rectangular}) \quad (12)$$

This expression is again of a form that does not involve the charge density. It is the result of applying the definition of divergence (11) to a differential volume element in *rectangular coordinates*.

If a differential volume unit $\rho \, d\rho \, d\phi \, dz$ in cylindrical coordinates, or $r^2 \sin \theta \, dr \, d\theta \, d\phi$ in spherical coordinates, had been chosen, expressions for divergence involving the components of the vector in the particular coordinate system and involving partial derivatives with respect to the variables of that system would have been obtained. These expressions are obtained in Appendix A and are given here for convenience:

$$\text{div } \mathbf{D} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho D_\rho) + \frac{1}{\rho} \frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} \quad (\text{cylindrical}) \quad (13)$$

$$\text{div } \mathbf{D} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 D_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta D_\theta) + \frac{1}{r \sin \theta} \frac{\partial D_\phi}{\partial \phi} \quad (\text{spherical}) \quad (14)$$

These relationships are also shown at the end of this book for easy reference.

It should be noted that the divergence is an operation which is performed on a vector, but that the result is a scalar. We should recall that, in a somewhat similar way, the dot or scalar product was a multiplication of two vectors which yielded a scalar.

For some reason, it is a common mistake on meeting divergence for the first time to impart a vector quality to the operation by scattering unit vectors around in

³ Having chosen a differential element of volume within the water, the gradual decrease in water level with time will eventually cause the volume element to lie above the surface of the water. At the instant the surface of the water intersects the volume element, the divergence is positive and the small volume is a source. This complication is avoided above by specifying an integral point.

the partial derivatives. Divergence merely tells us *how much* flux is leaving a small volume on a per-unit-volume basis; no direction is associated with it.

We can illustrate the concept of divergence by continuing with the example at the end of Section 3.4.

EXAMPLE 3.4

Find $\text{div } \mathbf{D}$ at the origin if $\mathbf{D} = e^{-x} \sin y \mathbf{a}_x - e^{-x} \cos y \mathbf{a}_y + 2z \mathbf{a}_z$.

Solution. We use (10) to obtain

$$\begin{aligned}\text{div } \mathbf{D} &= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \\ &= -e^{-x} \sin y + e^{-x} \sin y + 2 = 2\end{aligned}$$

The value is the constant 2, regardless of location.

If the units of \mathbf{D} are C/m^2 , then the units of $\text{div } \mathbf{D}$ are C/m^3 . This is a volume charge density, a concept discussed in the next section.

D3.7. In each of the following parts, find a numerical value for $\text{div } \mathbf{D}$ at the point specified: (a) $\mathbf{D} = (2xyz - y^2)\mathbf{a}_x + (x^2z - 2xy)\mathbf{a}_y + x^2y\mathbf{a}_z$ C/m^2 at $P_A(2, 3, -1)$; (b) $\mathbf{D} = 2\rho z^2 \sin^2 \phi \mathbf{a}_\rho + \rho z^2 \sin 2\phi \mathbf{a}_\phi + 2\rho^2 z \sin^2 \phi \mathbf{a}_z$ C/m^2 at $P_B(\rho = 2, \phi = 110^\circ, z = -1)$; (c) $\mathbf{D} = 2r \sin \theta \cos \phi \mathbf{a}_r + r \cos \theta \cos \phi \mathbf{a}_\theta - r \sin \phi \mathbf{a}_\phi$ C/m^2 at $P_C(r = 1.5, \theta = 30^\circ, \phi = 50^\circ)$.

Ans. (a) -10.00 ; (b) 9.06 ; (c) 1.29

3.4.3 Maxwell's First Equation: Gauss's Law in Point Form

Finally, we can combine Eqs. (9) and (12) and form the relation between electric flux density and charge density:

$$\boxed{\text{div } \mathbf{D} = \rho_v} \quad (15)$$

This is the first of Maxwell's four equations as they apply to electrostatics and steady magnetic fields, and it states that the electric flux per unit volume leaving a vanishingly small volume unit is exactly equal to the volume charge density there. This equation is aptly called the *point form of Gauss's law*. Gauss's law relates the flux leaving any closed surface to the charge enclosed, and Maxwell's first equation makes an identical statement on a per-unit-volume basis for a vanishingly small volume, or at a point. Because the divergence may be expressed as the sum of three partial derivatives, Maxwell's first equation is also described as the differential-equation form of Gauss's law, and conversely, Gauss's law is recognized as the integral form of Maxwell's first equation.

As a specific illustration, let us consider the divergence of \mathbf{D} in the region about a point charge Q located at the origin. We have the field

$$\mathbf{D} = \frac{Q}{4\pi r^2} \mathbf{a}_r$$

and use (14), the expression for divergence in spherical coordinates:

$$\text{div } \mathbf{D} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 D_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (D_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial D_\phi}{\partial \phi}$$

Because D_θ and D_ϕ are zero, we have

$$\text{div } \mathbf{D} = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{Q}{4\pi r^2} \right) = 0 \quad (\text{if } r \neq 0)$$

Thus, $\rho_v = 0$ everywhere except at the origin, where it is infinite.

The divergence operation is not limited to electric flux density; it can be applied to any vector field. We will apply it to several other electromagnetic fields in the coming chapters.

D3.8. Determine an expression for the volume charge density associated with each \mathbf{D} field: (a) $\mathbf{D} = \frac{4xy}{z} \mathbf{a}_x + \frac{2x^2}{z} \mathbf{a}_y - \frac{2x^2y}{z^2} \mathbf{a}_z$; (b) $\mathbf{D} = z \sin \phi \mathbf{a}_\rho + z \cos \phi \mathbf{a}_\phi + \rho \sin \phi \mathbf{a}_z$; (c) $\mathbf{D} = \sin \theta \sin \phi \mathbf{a}_r + \cos \theta \sin \phi \mathbf{a}_\theta + \cos \phi \mathbf{a}_\phi$.

Ans. $\frac{4y}{z^3} (x^2 + z^2)$; 0; 0.

3.5 DIVERGENCE THEOREM

Gauss's law for the electric field as we have used it is a specialization of what is called the *divergence theorem* in field theory. This general theorem is applied in other ways to different problems in physics, as well as to a few more in electromagnetics. In this section, we develop this broader perspective and demonstrate the use of the theorem in simplifying some otherwise complicated problems.

3.5.1 The Del Operator

As divergence is an operation on a vector yielding a scalar result, just as the dot product of two vectors gives a scalar result, it seems possible that we can find something that may be dotted formally with \mathbf{D} to yield the scalar

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

Obviously, this cannot be accomplished by using a dot *product*; the process must be a dot *operation*.

With this in mind, we define the *del operator* ∇ as a *vector operator*,

$$\nabla = \frac{\partial}{\partial x} \mathbf{a}_x + \frac{\partial}{\partial y} \mathbf{a}_y + \frac{\partial}{\partial z} \mathbf{a}_z \quad (16)$$

Similar *scalar operators* appear in several methods of solving differential equations where we often let D replace d/dx , D^2 replace d^2/dx^2 , and so forth.⁴ ∇ is treated in every way as an ordinary vector with the one important exception that partial derivatives result instead of products of scalars.

3.5.2 Obtaining Divergence with the Del Operator

Consider the operation $\nabla \cdot \mathbf{D}$, signifying

$$\nabla \cdot \mathbf{D} = \left(\frac{\partial}{\partial x} \mathbf{a}_x + \frac{\partial}{\partial y} \mathbf{a}_y + \frac{\partial}{\partial z} \mathbf{a}_z \right) \cdot (D_x \mathbf{a}_x + D_y \mathbf{a}_y + D_z \mathbf{a}_z)$$

We first consider the dot products of the unit vectors, discarding the six zero terms, and obtain the result that we recognize as the divergence of \mathbf{D} :

$$\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \text{div}(\mathbf{D})$$

The use of $\nabla \cdot \mathbf{D}$ is much more prevalent than that of $\text{div } \mathbf{D}$, although both usages have their advantages. Writing $\nabla \cdot \mathbf{D}$ allows us to obtain simply and quickly the correct partial derivatives, but only in rectangular coordinates, as we will see. On the other hand, $\text{div } \mathbf{D}$ is an excellent reminder of the physical interpretation of divergence. We will use the operator notation $\nabla \cdot \mathbf{D}$ from now on to indicate the divergence operation.

The vector operator ∇ is used not only with divergence, but also with several other very important operations that appear later. One of these, the *gradient*, is ∇u , where u is any scalar field, and it leads to

$$\nabla u = \left(\frac{\partial}{\partial x} \mathbf{a}_x + \frac{\partial}{\partial y} \mathbf{a}_y + \frac{\partial}{\partial z} \mathbf{a}_z \right) u = \frac{\partial u}{\partial x} \mathbf{a}_x + \frac{\partial u}{\partial y} \mathbf{a}_y + \frac{\partial u}{\partial z} \mathbf{a}_z$$

The ∇ operator does not have a specific form in other coordinate systems. If we are considering \mathbf{D} in cylindrical coordinates, then $\nabla \cdot \mathbf{D}$ still indicates the divergence of \mathbf{D} , or

$$\nabla \cdot \mathbf{D} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho D_\rho) + \frac{1}{\rho} \frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

where this expression has been taken from Section 3.5. We have no form for ∇ itself to help us obtain this sum of partial derivatives. This means that ∇u , as yet unnamed but easily written in rectangular coordinates, cannot be expressed by us at this time in cylindrical coordinates. Such an expression will be obtained when ∇u is defined in Chapter 4.

⁴This scalar operator D , which will not appear again, is not to be confused with the electric flux density.

3.5.3 Divergence Theorem

We close the treatment of divergence by presenting a theorem that brings the discussion full circle, the *divergence theorem*. This theorem applies to any vector field for which the appropriate partial derivatives exist, although it is easiest for us to develop it for the electric flux density. We have actually obtained it already and now have little more to do than point it out and name it, for starting from Gauss's law, we have

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \int_{\text{vol}} \rho_v dv = \int_{\text{vol}} \nabla \cdot \mathbf{D} dv$$

The first and last expressions constitute the divergence theorem,

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{vol}} \nabla \cdot \mathbf{D} dv} \quad (17)$$

which may be stated as follows:

The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface.

The divergence theorem is also known as *Gauss's theorem*, and in fact Gauss's law as we have used it is nothing more than an application of the divergence theorem to electrostatics. Again, we emphasize that the theorem is true for any vector field, and we will have occasion later to apply it to several different fields. Its benefits derive from the fact that it relates a triple integration *throughout some volume* to a double integration *over the surface* of that volume. For example, it is much easier to look for leaks in a bottle full of some agitated liquid by inspecting the surface than by calculating the velocity at every internal point. It should also be pointed out that Eq. (17), as applied to the electric flux density, is Maxwell's first equation *in integral form*.

The divergence theorem becomes obvious physically if we consider a volume v , shown in cross section in Figure 3.7, which is surrounded by a closed surface S .

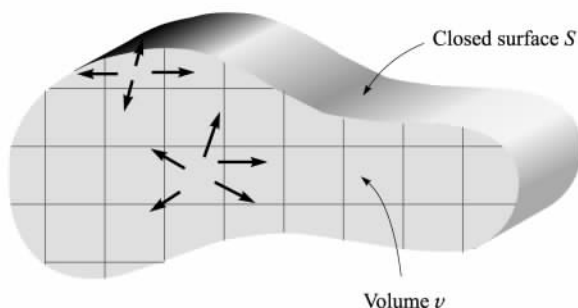


Figure 3.7 The divergence theorem states that the total flux crossing the closed surface is equal to the integral of the divergence of the flux density throughout the enclosed volume. The volume is shown here in cross section.

Division of the volume into a number of small compartments of differential size and consideration of one cell show that the flux diverging from such a cell *enters*, or *converges* on, the adjacent cells unless the cell contains a portion of the outer surface. In summary, the divergence of the flux density throughout a volume leads, then, to the same result as determining the net flux crossing the enclosing surface.

EXAMPLE 3.5

Evaluate both sides of the divergence theorem for the field $\mathbf{D} = 2xy\mathbf{a}_x + x^2\mathbf{a}_y$ C/m² and the rectangular parallelepiped formed by the planes $x = 0$ and 1 , $y = 0$ and 2 , and $z = 0$ and 3 .

Solution. Evaluating the surface integral first, we note that \mathbf{D} is parallel to the surfaces at $z = 0$ and $z = 3$, so $\mathbf{D} \cdot d\mathbf{S} = 0$ there. For the remaining four surfaces we have

$$\begin{aligned}\oint_S \mathbf{D} \cdot d\mathbf{S} &= \int_0^3 \int_0^2 (\mathbf{D})_{x=0} \cdot (-dy \, dz \, \mathbf{a}_x) + \int_0^3 \int_0^2 (\mathbf{D})_{x=1} \cdot (dy \, dz \, \mathbf{a}_x) \\ &\quad + \int_0^3 \int_0^1 (\mathbf{D})_{y=0} \cdot (-dx \, dz \, \mathbf{a}_y) + \int_0^3 \int_0^1 (\mathbf{D})_{y=2} \cdot (dx \, dz \, \mathbf{a}_y) \\ &= -\int_0^3 \int_0^2 (D_x)_{x=0} dy \, dz + \int_0^3 \int_0^2 (D_x)_{x=1} dy \, dz \\ &\quad - \int_0^3 \int_0^1 (D_y)_{y=0} dx \, dz + \int_0^3 \int_0^1 (D_y)_{y=2} dx \, dz\end{aligned}$$

However, $(D_x)_{x=0} = 0$, and $(D_y)_{y=0} = (D_y)_{y=2}$, which leaves only

$$\begin{aligned}\oint_S \mathbf{D} \cdot d\mathbf{S} &= \int_0^3 \int_0^2 (D_x)_{x=1} dy \, dz = \int_0^3 \int_0^2 2y \, dy \, dz \\ &= \int_0^3 4 \, dz = 12\end{aligned}$$

Since

$$\nabla \cdot \mathbf{D} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(x^2) = 2y$$

the volume integral becomes

$$\begin{aligned}\int_{\text{vol}} \nabla \cdot \mathbf{D} \, dv &= \int_0^3 \int_0^2 \int_0^1 2y \, dx \, dy \, dz = \int_0^3 \int_0^2 2y \, dy \, dz \\ &= \int_0^3 4 \, dz = 12\end{aligned}$$

and the check is accomplished. Remembering Gauss's law, we see that we have also determined that a total charge of 12 C lies within this parallelepiped.

D3.9. Given the field $\mathbf{D} = 6\rho \sin \frac{1}{2}\phi \, \mathbf{a}_\rho + 1.5\rho \cos \frac{1}{2}\phi \, \mathbf{a}_\phi$ C/m², evaluate both sides of the divergence theorem for the region bounded by $\rho = 2$, $\phi = 0$, $\phi = \pi$, $z = 0$, and $z = 5$.

Ans. 225; 225

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3. Plonus, M. A. *Applied Electromagnetics*. New York: McGraw-Hill, 1978. This book contains rather detailed descriptions of many practical devices that illustrate electromagnetic applications. For example, see the discussion of xerography on pp. 95–98 as an electrostatics application.
4. Skilling, H. H. *Fundamentals of Electric Waves*. 2d ed. New York: John Wiley & Sons, 1948. The operations of vector calculus are well illustrated. Divergence is discussed on pp. 22 and 38. Chapter 1 is interesting reading.
5. Thomas, G. B., Jr., and R. L. Finney. (See Suggested References for Chapter 1.) The divergence theorem is developed and illustrated from several different points of view on pp. 976–980.

CHAPTER 3 PROBLEMS



- 3.1** Suppose that the Faraday concentric sphere experiment is performed in free space using a central charge at the origin, Q_1 , and with hemispheres of radius a . A second charge Q_2 (this time a point charge) is located at distance R from Q_1 , where $R \gg a$. (a) What is the force on the point charge before the hemispheres are assembled around Q_1 ? (b) What is the force on the point charge after the hemispheres are assembled but before they are discharged? (c) What is the force on the point charge after the hemispheres are assembled and after they are discharged? (d) Qualitatively, describe what happens as Q_2 is moved toward the sphere assembly to the extent that the condition $R \gg a$ is no longer valid.
- 3.2** An electric field in free space is $\mathbf{E} = (5z^2/\epsilon_0)\hat{\mathbf{a}}_z$ V/m. Find the total charge contained within a cube, centered at the origin, of 4-m side length, in which all sides are parallel to coordinate axes (and therefore each side intersects an axis at ± 2).
- 3.3** Consider an electric dipole in free space, consisting of point charge q at location $z = +d/2$, and point charge $-q$ at location $z = -d/2$. The electric field intensity in the xy plane is (see Problem 2.7):

$$\mathbf{E} = \frac{-qd \mathbf{a}_z}{4\pi\epsilon_0[\rho^2 + (d/2)^2]^{3/2}}$$

where ρ is the radius from the origin in cylindrical coordinates.

- (a) Determine the net electric flux associated with this field that penetrates the xy plane. (b) Interpret your result as it relates to Gauss's law.

- 3.4 † An electric field in free space is $\mathbf{E} = (5z^3/\epsilon_0) \hat{\mathbf{a}}_z$ V/m. Find the total charge contained within a sphere of 3-m radius, centered at the origin.

- 3.5 † A volume charge distribution in free space is characterized by the density

$$\rho_v = \frac{q}{2Ad} \exp(-|z|/d)$$

where d is a distance along z , A is the area of a surface parallel to the xy plane, and q is a fixed charge quantity. The charge distribution exists everywhere. (a) Find the electric field intensity, \mathbf{E} , everywhere. (b) What is the interpretation of q ?

- 3.6 † In free space, a volume charge of constant density $\rho_v = \rho_0$ exists within the region $-\infty < x < \infty$, $-\infty < y < \infty$, and $-d/2 < z < d/2$. Find \mathbf{D} and \mathbf{E} everywhere.

- 3.7 † A spherically symmetric charge distribution in free space is characterized by the charge density

$$\rho_v = \frac{qb}{r^2} \exp(-br) \quad \text{C/m}^3 \quad (0 < r < \infty)$$

(a) Find the electric field intensity, $\mathbf{E}(r)$, everywhere. (b) Find the total charge present.

- 3.8 † Use Gauss's law in integral form to show that an inverse distance field in spherical coordinates, $\mathbf{D} = A\mathbf{a}_r/r$, where A is a constant, requires every spherical shell of 1 m thickness to contain $4\pi A$ coulombs of charge. Does this indicate a continuous charge distribution? If so, find the charge density variation with r .

- 3.9 † A sphere of radius a in free space contains charge of density $\rho_v = \rho_0 r/a$, where ρ_0 is a constant. (a) Find the electric field intensity, \mathbf{E}_I , inside the sphere. (b) Find the electric field intensity, \mathbf{E}_{II} , outside the sphere. (c) A spherical shell of radius b is positioned concentrically around the sphere. What surface charge density, ρ_s , must exist on the shell so that the electric field at locations $r > b$ is zero? (d) What electrostatic force per unit area is exerted by the solid sphere on the spherical shell?

- 3.10 † An infinitely long cylindrical dielectric of radius b contains charge within its volume of density $\rho_v = a\rho^2$, where a is a constant. Find the electric field strength, \mathbf{E} , both inside and outside the cylinder.

- 3.11 † Consider a cylindrical charge distribution having infinite length in z , but which has a radial dependence in charge density given by the gaussian, $\rho_v(\rho) = \rho_0 \exp[-(\rho/b)^2]$. (a) Find the electric field intensity, \mathbf{E} , at large radii, $\rho \gg b$. This enables the enclosed charge integral in Gauss's law to be approximated using an infinite upper limit in radius. (b) Compare your result to the field outside a charged cylinder of radius b containing uniform charge density ρ_0 .

- 3.12 † The sun radiates a total power of about 3.86×10^{26} watts (W). If we imagine the sun's surface to be marked off in latitude and longitude and

assume uniform radiation, (a) what power is radiated by the region lying between latitude 50° N and 60° N and longitude 12° W and 27° W?

(b) What is the power density on a spherical surface 93,000,000 miles from the sun in W/m^2 ?

- 3.13** Spherical surfaces at $r = 2, 4$, and 6 m carry uniform surface charge densities of 20 nC/m^2 , -4 nC/m^2 , and ρ_{s0} , respectively. (a) Find \mathbf{D} at $r = 1, 3$, and 5 m. (b) Determine ρ_{s0} such that $\mathbf{D} = 0$ at $r = 7$ m.
- 3.14** A certain light-emitting diode (LED) is centered at the origin with its surface in the xy plane. At far distances, the LED appears as a point, but the glowing surface geometry produces a far-field radiation pattern that follows a raised cosine law: that is, the optical power (flux) density in W/m^2 is given in spherical coordinates by

$$P_d = P_0 \frac{\cos^2 \theta}{2\pi r^2} \mathbf{a}_r \quad \text{W/m}^2$$

where θ is the angle measured with respect to the direction that is normal to the LED surface (in this case, the z axis), and r is the radial distance from the origin at which the power is detected. (a) In terms of P_0 , find the total power in watts emitted in the upper half-space by the LED. (b) Find the cone angle, θ_1 , within which half the total power is radiated, that is, within the range $0 < \theta < \theta_1$. (c) An optical detector, having a 1-mm^2 cross-sectional area, is positioned at $r = 1$ m and at $\theta = 45^\circ$, such that it faces the LED. If one nanowatt is measured by the detector, what (to a very good estimate) is the value of P_0 ?

- 3.15** Volume charge density is located as follows: $\rho_v = 0$ for $\rho < 1$ mm and for $\rho > 2$ mm, $\rho_v = 4\rho \text{ } \mu\text{C/m}^3$ for $1 < \rho < 2$ mm. (a) Calculate the total charge in the region $0 < \rho < \rho_1$, $0 < z < L$, where $1 < \rho_1 < 2$ mm. (b) Use Gauss's law to determine D_ρ at $\rho = \rho_1$. (c) Evaluate D_ρ at $\rho = 0.8$ mm, 1.6 mm, and 2.4 mm.
- 3.16** An electric flux density is given by $\mathbf{D} = D_0 \mathbf{a}_\rho$, where D_0 is a given constant. (a) What charge density generates this field? (b) For the specified field, what total charge is contained within a cylinder of radius a and height b , where the cylinder axis is the z axis?
- 3.17** In a region having spherical symmetry, volume charge is distributed according to:

$$\rho_v(r) = \rho_0 \frac{\sin(\pi r/a)}{r^2} \quad \text{C/m}^3$$

Find the surfaces on which $\mathbf{E} = 0$.

- 3.18** State whether the divergence of the following vector fields is positive, negative, or zero: (a) the thermal energy flow in $\text{J}/(\text{m}^2 \cdot \text{s})$ at any point in a freezing ice cube; (b) the current density in A/m^2 in a bus bar carrying direct

current; (c) the mass flow rate in $\text{kg}/(\text{m}^2 \cdot \text{s})$ below the surface of water in a basin, in which the water is circulating clockwise as viewed from above.

- 3.19** A spherical surface of radius 3 mm is centered at $P(4, 1, 5)$ in free space. Let $\mathbf{D} = x\mathbf{a}_x \text{ C/m}^2$. Use the results of Section 3.4 to estimate the net electric flux leaving the spherical surface.
- 3.20** A radial electric field distribution in free space is given in spherical coordinates as:

$$\begin{aligned}\mathbf{E}_1 &= \frac{r\rho_0}{3\epsilon_0} \mathbf{a}_r & (r \leq a) \\ \mathbf{E}_2 &= \frac{(2a^3 - r^3)\rho_0}{3\epsilon_0 r^2} \mathbf{a}_r & (a \leq r \leq b) \\ \mathbf{E}_3 &= \frac{(2a^3 - b^3)\rho_0}{3\epsilon_0 r^2} \mathbf{a}_r & (r \geq b)\end{aligned}$$

where ρ_0 , a , and b are constants. (a) Determine the volume charge density in the entire region ($0 \leq r \leq \infty$) by the appropriate use of $\nabla \cdot \mathbf{D} = \rho_v$. (b) In terms of given parameters, find the total charge, Q , within a sphere of radius r where $r > b$.

- 3.21** In a region exhibiting spherical symmetry, electric flux density is found to be $\mathbf{D}_1 = \rho_0 r/3 \mathbf{a}_r$ ($0 < r < a$), $\mathbf{D}_2 = 0$ ($a < r < b$), and $\mathbf{D}_3 = (a^3 \rho_0)/(3r^2) \mathbf{a}_r$ ($r < b$). (a) Find the charge configuration that would produce the given field. (b) What total charge is present?
- 3.22** (a) A flux density field is given as $\mathbf{F}_1 = 5\mathbf{a}_z$. Evaluate the outward flux of \mathbf{F}_1 through the hemispherical surface, $r = a$, $0 < \theta < \pi/2$, $0 < \phi < 2\pi$. (b) What simple observation would have saved a lot of work in part a? (c) Now suppose the field is given by $\mathbf{F}_2 = 5z\mathbf{a}_z$. Using the appropriate surface integrals, evaluate the net outward flux of \mathbf{F}_2 through the closed surface consisting of the hemisphere of part a and its circular base in the xy plane. (d) Repeat part c by using the divergence theorem and an appropriate volume integral.
- 3.23** (a) A point charge Q lies at the origin. Show that $\text{div } \mathbf{D}$ is zero everywhere except at the origin. (b) Replace the point charge with a uniform volume charge density ρ_{v0} for $0 < r < a$. Relate ρ_{v0} to Q and a so that the total charge is the same. Find $\text{div } \mathbf{D}$ everywhere.
- 3.24** In a region in free space, electric flux density is found to be

$$\mathbf{D} = \begin{cases} \rho_0(z + 2d)\mathbf{a}_z \text{ C/m}^2 & (-2d \leq z \leq 0) \\ -\rho_0(z - 2d)\mathbf{a}_z \text{ C/m}^2 & (0 \leq z \leq 2d) \end{cases}$$

Everywhere else, $\mathbf{D} = 0$. (a) Using $\nabla \cdot \mathbf{D} = \rho_v$, find the volume charge density as a function of position everywhere. (b) Determine the electric

flux that passes through the surface defined by $z = 0$, $-a \leq x \leq a$, $-b \leq y \leq b$. (c) Determine the total charge contained within the region $-a \leq x \leq a$, $-b \leq y \leq b$, $-d \leq z \leq d$. (d) Determine the total charge contained within the region $-a \leq x \leq a$, $-b \leq y \leq b$, $0 \leq z \leq 2d$.

- 3.25** Within the spherical shell, $3 < r < 4$ m, the electric flux density is given as $\mathbf{D} = 5(r - 3)^3 \mathbf{a}_r$ C/m². (a) What is the volume charge density at $r = 4$? (b) What is the electric flux density at $r = 4$? (c) How much electric flux leaves the sphere $r = 4$? (d) How much charge is contained within the sphere $r = 4$?
- 3.26** If we have a perfect gas of mass density ρ_m kg/m³, and we assign a velocity \mathbf{U} m/s to each differential element, then the mass flow rate is $\rho_m \mathbf{U}$ kg/(m² · s). Physical reasoning then leads to the *continuity equation*, $\nabla \cdot (\rho_m \mathbf{U}) = -\partial \rho_m / \partial t$. (a) Explain in words the physical interpretation of this equation. (b) Show that $\oint_S \rho_m \mathbf{U} \cdot d\mathbf{S} = -dM/dt$, where M is the total mass of the gas within the constant closed surface S , and explain the physical significance of the equation.
- 3.27** Consider a slab of material containing a volume charge distribution throughout. The slab is of length d in the z direction, and its dimensions in x and y represent a cross-sectional area of A . Free space permittivity exists throughout. The electric field in the slab is given by

$$\mathbf{E} = \frac{\rho_0}{\epsilon_0 \alpha} \exp(-\alpha z) \mathbf{a}_z \quad \text{V/m}$$

where ρ_0 is a positive constant. (a) Find the volume charge density ρ_v in the slab. (b) Find the total charge in the slab. (c) Verify your result for part *b* by evaluating the net outward flux of \mathbf{D} through the slab surfaces.

- 3.28** Repeat Problem 3.8, but use $\nabla \cdot \mathbf{D} = \rho_v$ and take an appropriate volume integral.
- 3.29** In the region of free space that includes the volume $2 < x$, y , $z < 3$, $\mathbf{D} = \frac{2}{z^2} (yz \mathbf{a}_x + xz \mathbf{a}_y - 2xy \mathbf{a}_z)$ C/m². (a) Evaluate the volume integral side of the divergence theorem for the volume defined here. (b) Evaluate the surface integral side for the corresponding closed surface.
- 3.30** (a) Use Maxwell's first equation, $\nabla \cdot \mathbf{D} = \rho_v$, to describe the variation of the electric field intensity with x in a region in which no charge density exists and in which a nonhomogeneous dielectric has a permittivity that increases exponentially with x . The field has an x component only; (b) repeat part *a*, but with a radially directed electric field (spherical coordinates) in which again $\rho_v = 0$, but in which the permittivity *decreases* exponentially with r .
- 3.31** Given the flux density $\mathbf{D} = \frac{16}{r} \cos(2\theta) \mathbf{a}_\theta$ C/m², use two different methods to find the total charge within the region $1 < r < 2$ m, $1 < \theta < 2$ rad, $1 < \phi < 2$ rad.