第1篇

Part Ⅰ

数理逻辑

Mathematical Logic

Chapter 1　Propositional Logic

Propositional logic can also be called propositional calculus or statement calculus. It mainly studies the deducible relationship between the premise and the conclusion which is composed of the proposition as the basic unit. So what is a proposition? How to express a proposition? How to derive some conclusions from a set of premises? We will take you to discuss these core issues in detail in the following.

1.1　Propositions and Connectives

A proposition is expressed as a statement with a definite meaning. A proposition always has a value called truth value. There are only "True" and "False" values, which are called True and False, usually with 0 or F for "False" and 1 or T for "True".

Statement that represents a proposition must have two conditions: first, the statement is a declarative sentence; second, the statement has a uniquely determined true or false meaning. There are two steps to judge whether a given sentence is a proposition: first, whether it is a declarative sentence; second, whether it has a unique truth value.

There are two types of propositions: the first type is the proposition that cannot be decomposed into simpler statements, which is called atomic proposition; the second type of proposition consists of a combination of connectives, punctuation marks and atomic propositions.

All of these propositions should have certain truth values.

The atomic proposition, also known as a simple proposition, is an inference about a single property of an object, and it can also represent any true or false statement about a certain object or situation. Its semantic truth value is determined by objective facts.

Compound propositions, which can be decomposed into some simple propositions in grammatical structure, are formed by some simple propositions constructed by proposition connectors.

An example is given to illustrate the concept of the proposition.

(1) The sun rises in the east and sets in the west.

(2) The grass is green.

(3) Stand up!

(4) 71% of the earth's surface is water.

(5) Will it rain tomorrow?

(6) What a beautiful day!

(7) In the afternoon I go to the library, or to the laboratory.

(8) Both Zhang San and Li Si are postgraduates of Jiangsu University.

In the above examples, (1), (2), (4), (7), (8) are propositions. Among them, (1), (7), (8) are compound propositions. (3), (5), (6) are not propositions, because they aren't declarative sentences.

In mathematical logic, compound propositions are usually constituted by "connectives". Propositional connectives also known as propositional operators, have strict logical meanings to ensure the consistency between the semantics of the symbolic system and those of the original natural system. There are five logical connectives in common use: Negation Connective, Conjunction Connective, Disjunction Connective, Conditional Connective and Biconditional Connective.

1.1.1 Negation Connective

Definition 1.1.1 If P is a proposition, then the negation of P is a compound proposition, denoted as "$\neg P$" and pronounced as "not P" or "P's negation". If P is T, then $\neg P$ is F; If P is F, then the truth value of $\neg P$ is T.

The connective "$\neg$" can also be seen as a logical operator, which is a unary operation. The relationship between P and $\neg P$ is shown in Table 1.1.1, which is called the truth table of the negation connective "$\neg$".

Example 1.1.1 Negate the following propositions.

P: Zhang San is a college student.

$\neg P$: Zhang San is not a college student.

1.1.2 Conjunction Connective

Definition 1.1.2 If P and Q are both propositions, then the conjunction of P and Q is a compound proposition, denoted as $P \wedge Q$, pronounced as "P and Q" or "P conjunction Q". The value of $P \wedge Q$ is T if and only if both P and Q are T.

The connective "$\wedge$" can also be viewed as a binary logical operation. The truth value table of the connective "$\wedge$" is shown in Table 1.1.2.

Table 1.1.1

P	$\neg P$
F	T
T	F

Table 1.1.2

P	Q	$P \wedge Q$
F	F	F
F	T	F
T	F	F
T	T	T

Example 1.1.2 Suppose P: The COVID-19 epidemic was raging in 2020. Q: The COVID-19 broke out in many countries around the world. Then $P \wedge Q$: The COVID-19 epidemic was raging and broke out in many countries worldwide in 2020.

1.1.3 Disjunction Connective

Definition 1.1.3 Suppose both P and Q are the propositions, and the disjunction of P and Q is a compound proposition, as $P \vee Q$, pronounced "P or Q," or "P disjunction Q". If and only if P and Q are F, $P \vee Q$ is F.

Connective "$\vee$" can also be seen as a logic operation, and it is a binary logic operation. "$\vee$" similar to "or" in Chinese, but not the same. In Chinese, there are "inclusive or" and "exclusive or". The truth table of the conjunction "$\vee$" is shown in Table 1.1.3.

Example 1.1.3 Which of the following two propositions is inclusive or? Which is exclusive or?

(1) Watch the acrobatics on TV or in the theater. (exclusive or)

(2) The light bulb or switch is out of order. (inclusive or, "$\vee$" means inclusive or)

1.1.4 Conditional Connective

Definition 1.1.4 Suppose both P and Q are propositions, and the conditional proposition is a compound proposition, denoted as $P \rightarrow Q$, read "if P, then Q". If and only if P is T and Q is F, $P \rightarrow Q$ is F. P is the antecedent of the conditional proposition $P \rightarrow Q$, and Q is the decedent of the conditional proposition $P \rightarrow Q$.

The connective "$\rightarrow$" can also be viewed as a logical operation, which is a binary logical operation. The truth value table of the connective "$\rightarrow$" is shown in Table 1.1.4.

Table 1.1.3

P	Q	$P \vee Q$
F	F	F
F	T	T
T	F	T
T	T	T

Table 1.1.4

P	Q	$P \rightarrow Q$
F	F	T
F	T	T
T	F	F
T	T	T

Example 1.1.4 P: Xiao Yu studies hard.

Q: Xiao Yu does well in his studies.

$P \rightarrow Q$: If Xiao Yu studies hard, he will get good grades.

The connective "$\rightarrow$" and the Chinese "if..., then..." are similar, but different.

1.1.5 Biconditional Connective

Definition 1.1.5 Set both P and Q as propositions, the complex proposition $P \leftrightarrow Q$ can be called dual condition, which can be pronounced as "P dual condition Q" or "P can be if and only Q". It can be T if and only if the true value of P and Q are the same.

The connective "$\leftrightarrow$" can also be understood as a logical operation, which is binary logic.

The truth value table of the connective "↔" is shown in Table 1.1.5.

Table 1.1.5

P	Q	$P\leftrightarrow Q$	P	Q	$P\leftrightarrow Q$
F	F	T	T	F	F
F	T	F	T	T	T

As described above, a biconditional connective represents a sufficient and necessary relationship and can be determined only according to the definition of a connective, regardless of its antecedes.

Example 1.1.5 Suppose P: Li Si is a merit student.

Q: Li Si is excellent in moral, intellectual and physical qualities.

$P\leftrightarrow Q$: Li Si is a merit student if and only if he is excellent in moral, intellectual and physical qualities.

1.2 Propositional Formula and Translation

It has been mentioned before that propositions that do not contain other propositions as part of them are called atomic propositions. At the same time, atomic propositions cannot contain any connectives. Then a proposition containing at least one conjunction is called a compound proposition.

Let P and Q be any two propositions, then $\neg P$, $P\wedge Q$, $(P\wedge Q)\rightarrow(P\vee Q)$, $\neg P\rightarrow(Q\leftrightarrow\neg P)$ are compound propositions.

If P and Q are propositional variables, then all the above formulas are called propositional formulas, also called well-formed formulas. P and Q are called the components of a propositional formula.

Note: Propositional formulas contain propositional variables, so it is impossible to calculate their true values. Only when the propositional variables in a propositional formula are brought in with propositions that determine the truth values, a proposition can be obtained.

Definition 1.2.1 The symbol string formed according to the following rules is called the well－formed formulas of propositional calculus (wff), also known as propositional formula, or formula for short:

(1) A single propositional variable or constant is a well-formed formula.

(2) If A is a well-formed formula, then $\neg A$ is also a well-formed formula.

(3) If A and B are well-formed formulas, then $(A\wedge B)$, $(A\vee B)$, $(A\rightarrow B)$, $(A\leftrightarrow B)$ are well-formed formulas.

(4) If and only if (1), (2) and (3) are applied a finite number times, the resulting sign string is a well-formed formula.

The propositional formula is generally represented by the capital letters A, B, C, … .

The method of defining the formula is called inductive definition, which consists of three parts: basis, induction and boundary. Among them, (1) is called basis, (2) (3) is called induction, and(4) is called boundary.

For convenience, the involution formula is agreed as follows:

(1) To reduce the number of parentheses used, it is agreed that the outermost parentheses can be omitted.

(2) The order of priority of connectives is as follows: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$. According to this priority, if the brackets are removed and the operation order of the original formula can be omitted, then $P \wedge Q \rightarrow R$ is equivalent to$(P \wedge Q) \rightarrow R$, which is also a compound formula.

Example 1.2.1 Determine whether the following symbol string is a compound formula.

(1) $\neg (P \vee Q)$

(2) $(\neg P \rightarrow Q)$

(3) $(P \rightarrow (P \leftrightarrow Q))$

(4) $(P \leftrightarrow Q) \rightarrow (\vee Q)$

(5) $(P \wedge Q, (P \wedge Q) \wedge Q)$

Solution It can be seen from the above definitions that(1),(2),(3) are all well-formed formulas;(4) and(5) are not well-formed formulas. Because $\vee$ in(4) is binocular conjunction and a propositional argument is missing on the left; while there is a comma in(5), which does not meet the definition of the formula.

With the concept of well-formed formulas of connectives, well-formed formulas can be represented by compound formulas. This process is often called the symbolization of propositions. The whole process is equivalent to translating some sentences(propositions) in natural language into symbolic forms in mathematical logic. It can be carried out according to the following steps:

(1) Determines whether a given statement is a proposition.

(2) Find out the atomic propositions in the statement and determine the conjunctions corresponding to the conjunctions.

(3) In with or by grammar, the original proposition is expressed as a compound formula composed of the atomic proposition, connectives and parentheses.

Example 1.2.2 Symbolize the following proposition: The G124 train from Changsha to Zhenjiang leaves at 9:30 or 10:00 a.m.

Solution P: The G124 train from Changsha to Zhenjiang leaves at 9:30 a.m.

Q: The G124 train from Changsha to Zhenjiang leaves at 10:00 a.m.

In this example, the word "or" is exclusive or, which means is, the train can not leave at 9:30 and 10:00 a.m. at the same time, it can only be one of the two; while $\vee$ in mathematical logic is inclusive or, so we can't use $\vee$ directly. Then construct a truth table for analysis, as shown in Table 1.2.1.

It can be seen from the table the original proposition cannot, denoted by the five connectives separately, but can use the proposition and connectives, the propositional symbol can be turned into: $\neg(P\leftrightarrow Q)$.

Table 1.2.1

P	Q	original proposition	$P\leftrightarrow Q$	$\neg(P\leftrightarrow Q)$	P	Q	original proposition	$P\leftrightarrow Q$	$\neg(P\leftrightarrow Q)$
T	T	F	T	F	F	T	T	F	T
T	F	T	F	T	F	F	F	T	F

Example 1.2.3 Symbolizing the following proposition: He is not only gifted but also hardworking.

Solution P: He is gifted. Q: He works hard.

According to "not only ... but also ...", we can know the result of this question is $P \wedge Q$.

Example 1.2.4 Symbolize the following proposition: He is gifted but not hard working.

Solution The actual meaning of this sentence is that he is gifted and not hard working. So if the hypothesis is as follows:

P: He is gifted. $\neg Q$: He is not hard working.

The result of this question is $P \wedge \neg Q$.

Example 1.2.5 Symbolize the following proposition: unless you study hard, you will fail.

Solution The actual meaning of this sentence is that if you don't work hard, you'll lose the course. So if the hypothesis is as follows:

P: You work hard. Q: you will fail.

The result of this question is $\neg P \rightarrow Q$.

Example 1.2.6 Symbolizing the following propositions: He won the gold or silver award in the International Collegiate Programming Contest.

Solution Find out the atomic propositions and use propositional symbols to represent them:

A: He won the gold award in the International Collegiate Programming Contest.

B: He won the silver award in the International Collegiate Programming Contest.

So the result of this problem is $(A \wedge \neg B) \vee (\neg A \wedge B)$.

From the above examples, we can find that some connectives in natural language, such as "not only... but also..." "or" "although... but..." each have their specific meaning. Therefore, they need to be translated into appropriate logical connectives according to different situations. At the same time, to correctly express the relationship between propositions, sometimes we can list the "truth table" to further analyze each original proposition.

1.3 Truth Tables and Equivalent Formulas

From the above Example 1.2.2, when a proposition cannot be directly symbolized, the truth table is often used for auxiliary analysis, which will get the symbolic results faster. Therefore, we will further learn the related knowledge of the truth table.

Definition 1.3.1 Suppose P_1, P_2, …, P_n are all propositional variables appearing in proposition formula A. Assigning a true value to P_1, P_2, …, P_n is called an assignment or explanation of proposition formula A. If the assigned value makes the true value of A be T, then this assignment is called a real assignment of A. If the true value of A is set to F, then the assignment is called a false assignment of A.

Definition 1.3.2 In proposition formula A, a truth value of proposition formula A is determined by assigning a value to each value of proposition formula A. The truth value table of proposition formula A is formed by summarizing all truth values into a table.

Example 1.3.1 Construct the truth table of $P \vee \neg Q$.

Solution The results are shown in Table 1.3.1.

Example 1.3.2 Construct the truth table of $(P \wedge Q) \wedge \neg P$.

Solution The results are shown in Table 1.3.2.

Table 1.3.1

P	Q	$\neg Q$	$P \vee \neg Q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

Table 1.3.2

P	Q	$P \wedge Q$	$\neg P$	$(P \wedge Q) \wedge \neg P$
T	T	T	F	F
T	F	F	F	F
F	T	F	T	F
F	F	F	T	F

Example 1.3.3 Construct the truth table of $(P \vee Q) \vee (\neg P \wedge Q)$.

Solution The results are shown in Table 1.3.3.

Table 1.3.3

P	Q	$\neg P$	$P \vee Q$	$\neg P \wedge Q$	$(P \vee Q) \vee (\neg P \wedge Q)$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	F

Example 1.3.4 Construct the truth table of $\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)$.

Solution The results are shown in Table 1.3.4.

Table 1.3.4

P	Q	$P\wedge Q$	$\neg(P\wedge Q)$	$\neg P$	$\neg Q$	$\neg P\vee\neg Q$	$\neg(P\wedge Q)\leftrightarrow(\neg P\vee\neg Q)$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

In the truth table, the number of truth values of the propositional formula depends on the number of components. For example, there are four possible assignments for a propositional formula composed of two propositional variables, while there are eight possible assignments for a propositional formula composed of three propositional variables. Therefore, in general, there are 2^n truth value cases for propositional formulas composed of n propositional variables.

It can be found from the truth table that the corresponding truth values of some propositional formulas are the same as those of other propositional formulas under all assignments of components, the corresponding truth values of $\neg(Q\rightarrow P)$ and $\neg P\wedge Q$ are the same, as shown in Table 1.3.5.

Table 1.3.5

P	Q	$\neg P$	$Q\rightarrow P$	$\neg(Q\rightarrow P)$	$\neg P\wedge Q$
T	T	F	T	F	F
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	F	F

Definition 1.3.3 Given two propositional formulas A and B, let P_1, P_2, P_3, …, P_n be all atomic variables in A and B. If any group of truth values of P_1, P_2, P_3, …, P_n are assigned, and the true values of A and B are the same, then A and B are said to be equivalent or logically equal. Recorded as $A\Leftrightarrow B$.

It can be proved that propositional formula equivalence has the following three properties.

(1) Reflexivity, for any proposition formula A, $A\Leftrightarrow A$.

(2) Symmetry, for any propositional formula A and B, if $A\Leftrightarrow B$, then $B\Leftrightarrow A$.

(3) Transitivity, for any propositional formula A, B and C, if $A\Leftrightarrow B$, $B\Leftrightarrow C$, then$A\Leftrightarrow C$.

Example 1.3.5 Prove $P\leftrightarrow Q\Leftrightarrow(P\rightarrow Q)\wedge(Q\rightarrow P)$.

Proof It can be seen from Table 1.3.6 that $P\leftrightarrow Q$ and $P\rightarrow Q\wedge(Q\rightarrow P)$ have the same truth value, so the proposition is proved.

Table 1.3.6

P	Q	$P\rightarrow Q$	$Q\rightarrow P$	$P\leftrightarrow Q$	$(P\rightarrow Q)\wedge(Q\rightarrow P)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Example 1.3.6 Prove $P \vee (P \wedge Q) \Leftrightarrow P$, $P \wedge (P \vee Q) \Leftrightarrow P$.

Proof The truth table is shown in Table 1.3.7, and the equivalent relationship between the two formulas can be clearly seen from the truth table.

Table 1.3.7

P	Q	$P \wedge Q$	$P \vee Q$	$P \vee (P \wedge Q)$	$P \wedge (P \vee Q)$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	F	T	F	F
F	F	F	F	F	F

As shown in Table 1.3.8, common propositional equivalence theorems are listed.

Table 1.3.8

Number	The Law	Equivalence Relation
1	Involution law	$\neg\neg P \Leftrightarrow P$
2	Idempotent law	$P \vee P \Leftrightarrow P$, $P \wedge P \Leftrightarrow P$
3	Associative law	$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$, $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$
4	Exchange law	$P \vee Q \Leftrightarrow Q \vee P$, $P \wedge Q \Leftrightarrow Q \wedge P$
5	Distributive law	$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$, $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$
6	Absorption law	$P \vee (P \wedge Q) \Leftrightarrow P$, $P \wedge (P \vee Q) \Leftrightarrow P$
7	De Morgan's law	$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$, $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$
8	Identity law	$P \vee F \Leftrightarrow P$, $P \wedge T \Leftrightarrow P$
9	Zero law	$P \vee T \Leftrightarrow T$, $P \wedge F \Leftrightarrow F$
10	Negation law	$P \vee \neg P \Leftrightarrow T$, $P \wedge \neg P \Leftrightarrow F$
11	Conditional equivalence	$A \rightarrow B \Leftrightarrow \neg A \vee B$
12	Two conditional equivalence	$A \leftrightarrow B \Leftrightarrow (A \rightarrow B) \wedge (B \rightarrow A)$
13	Hypothetical translocation	$A \rightarrow B \Leftrightarrow \neg B \rightarrow \neg A$
14	Double conditional negation equivalence	$A \leftrightarrow B \Leftrightarrow \neg A \leftrightarrow \neg B$
15	Domestication of conjunctions	$\neg(A \leftrightarrow B) \Leftrightarrow A \leftrightarrow \neg B$

Example 1.3.7 Prove De Morgan's law $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$ by the truth table.

Proof Table 1.3.9 is the truth table of $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$. From the table, we can find that De Morgan's law is correct.

Table 1.3.9

P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

In a propositional formula, if a part of a proposition is replaced by a formula, a new formula will be produced. For example, if $(P \rightarrow Q)$ is replaced in $P \rightarrow (P \vee (P \rightarrow Q))$ with $(P \wedge Q)$, then $P \rightarrow (P \vee (P \wedge Q))$ is different from the original formula. To ensure that the