Chapter 3 Solution of Electrostatic Boundary Value Problems (静电场边界值问题求解)

3.1 Introduction(引言)

Electrostatic problems are those to find electric potential and/or electric field intensity due to static electric charges. In Chapter 2, several methods have been developed to find the electric potential and the electric field intensity when the charge distribution is known. In practical problems, however, the exact charge distribution is usually unknown, and as a result, the formulas in Chapter 2 cannot be applied directly. Instead, practical electrostatic problems might involve conducting bodies with given potentials, which can be modeled as a boundary-value problem in terms of the electric potential. In these cases, the electric fields can be found by solving a partial differential equation subject to the known boundary conditions on the surfaces of conducting bodies. Analytical solutions of the partial differential equation may be obtained if the electrostatic problem can be reduced to one-dimensional. For two-dimensional or three-dimensional problems, analytical solutions generally do not exist. Nevertheless, if the boundaries are of certain simple geometries, the method of images or the method of separation of variables can be used to provide analytical or semi-analytical solutions. ①

3.2 Poisson's and Laplace's Equations (泊松方程、拉普拉斯方程)

In Chapter 2, two fundamental equations governing the electrostatic fields are formulated as

$$\nabla\cdot \boldsymbol{D} = \rho_v \tag{3.1}$$

$$\nabla\times \boldsymbol{E} = \boldsymbol{0} \tag{3.2}$$

From (3.2), we introduced the electric potential φ that satisfies

$$\boldsymbol{E} = -\nabla\varphi \tag{3.3}$$

① 第2章给出了从已知的电荷分布出发求解电场的几种方法。然而在实际静电场问题中,电荷分布常常是未知的,而问题所在区域的边界上电位分布可能已知,这种情况下可以将静电场问题表述为关于电位的边界值问题,即给定边界条件,通过求解电位满足的二阶偏微分方程得到电位的解,然后对电位取负梯度,得到电场的解。本章从静电场电位满足的泊松/拉普拉斯方程出发,基于唯一性定理,介绍静电场边界值问题的几种典型的求解方法。

In a linear and isotropic medium, $\boldsymbol{D}=\varepsilon\boldsymbol{E}$. Therefore, (3.1) becomes

$$\nabla\cdot(\varepsilon\boldsymbol{E})=\rho_v \tag{3.4}$$

Substituting (3.3) into (3.4) leads to

$$\nabla\cdot(\varepsilon\nabla\varphi)=-\rho_v \tag{3.5}$$

where the permittivity ε can be a function of position. For a simple medium, ε is a constant and can be taken out of the divergence operation. Then we have

$$\nabla^2\varphi=-\frac{\rho_v}{\varepsilon} \tag{3.6}$$

where ∇^2 is the Laplacian operator as introduced in Section 1-12. (3.6) is known as **Poisson's equation** (泊松方程). It states that the Laplacian of φ equals $-\rho_v/\varepsilon$ for a simple medium where ρ_v is the volume density of free charges (which may be a function of space coordinates). If the charge distribution ρ_v is known everywhere in the entire free space, the solution of equation (3.6) is known as (2.58), which is rewritten as

$$\varphi(\boldsymbol{r})=\frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho_v(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}v' \tag{3.7}$$

However, in practical problems, the function ρ_v may not be known, or may be too complicated, which makes it difficult to evaluate the integration in (3.7). Then, instead of using (3.7), it is usually more practical to formulate the electrostatic problems as solving the Poisson's equation (3.6) subject to prescribed boundary conditions (e.g., given φ on certain conducting bodies). ①

Poisson's equation (3.6) is a second-order partial differential equation, which, in Cartesian coordinates, becomes

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=-\frac{\rho_v}{\varepsilon} \tag{3.8}$$

In cylindrical and spherical coordinates, the Poisson's equation becomes, respectively,

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\varphi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\varphi}{\partial\phi^2}+\frac{\partial^2\varphi}{\partial z^2}=-\frac{\rho_v}{\varepsilon} \tag{3.9}$$

and

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}=-\frac{\rho_v}{\varepsilon} \tag{3.10}$$

At points in a simple medium where there is no free charge, $\rho_v=0$ and the Poisson's equation (3.6) reduces to

$$\nabla^2\varphi=0 \tag{3.11}$$

which is known as **Laplace's equation** (拉普拉斯方程). Laplace's equation is the governing

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① 泊松方程(3.6)是根据静电场特性引入电位后,从静电场满足的基本方程直接推导得到的,也是静电场电位必须满足的基本方程。式(3.7)是泊松方程的解,然而,采用式(3.7)计算电位的前提条件是其中的电荷密度 ρ_v 在整个空间中已知,且积分区域 V' 包含所有的自由电荷。如果不知道电荷分布,无法直接采用式(3.7)得到静电场的解,但可以通过求解满足特定边界条件的泊松方程(3.6)得到静电场的解。

equation for many electrostatic problems involving a set of conductors maintained at given potentials. Once φ is found by solving the Laplace's equation, the electric field can be determined from $-\nabla\varphi$, and the charge distribution on the conductor surfaces can be determined from the boundary condition $\rho_s = \varepsilon E_n$. ①

Example 3-1 As shown in Figure 3-1, the potential difference across a parallel-plate capacitor is maintaned at U_0. The separation between the two plates of the capacitor is d. Assume the fringing effect can be neglected. Determine ① the potential distribution between the plates, and ② the surface charge densities on the plates.

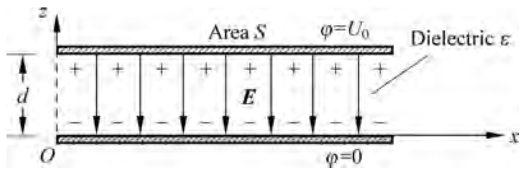

Figure 3-1 A parallel capacitor

Solution: This is essentially the same problem as Example 2-16. Now we solve it by solving the Laplace's equation satisfied by the electric potential since the charge density $\rho_v = 0$ between the plates.

(1) By ignoring the fringing effect of the electric field, we assume the field distribution is the same as if the plates were infinitely large. In other words, the potential φ has no variation in the x-and y-directions. Hence, Laplace's equation is then simplified to

$$\frac{d^2\varphi}{dz^2} = 0 \tag{3.12}$$

where d^2/dz^2 is used instead of $\partial^2/\partial z^2$ because z is the only variable in this problem. Integration of (3.12) with respect to z gives

$$\frac{d\varphi}{dz} = C_1$$

where C_1 is an unknown constant coefficient. Integrating again, we obtain

$$\varphi = C_1 z + C_2 \tag{3.13}$$

To determine the two unknown coefficients C_1 and C_2, we use the following two boundary conditions:

$$\text{At } z = 0, \quad \varphi = 0 \tag{3.14a}$$

$$\text{At } z = d, \quad \varphi = U_0 \tag{3.14b}$$

Substitution of (3.14a) and (3.14b) respectively into (3.13) yields two equations, from which the two unknown coefficients can be solved to obtain $C_1 = U_0/d$ and $C_2 = 0$. Hence the potential distribution between the plates is

① 在无源区域(即电荷密度为零的区域,或者说是电荷分布区域以外的空间),泊松方程变为拉普拉斯方程。在很多实际问题,包括以下几个例子中,感兴趣的电场都是分布在无源区域中,因此可以通过求解满足特定边界条件的拉普拉斯方程(3.11)得到静电场的解。

$$\varphi = \frac{U_0}{d}z \tag{3.15}$$

(2) The surface charge densities can be found by using the boundary condition of the $\boldsymbol{E}$ field on the surfaces of the conducting plates($z=0$ and $z=d$). We first find the $\boldsymbol{E}$ field by using (3.3):

$$\boldsymbol{E} = -\boldsymbol{e}_z \frac{d\varphi}{dz} = -\boldsymbol{e}_z \frac{U_0}{d}$$

Then the surface charge densities at the conducting plates are obtained as

$$\rho_s = \varepsilon \boldsymbol{e}_n \cdot \boldsymbol{E} = \varepsilon \boldsymbol{e}_n \cdot \left(-\boldsymbol{e}_z \frac{U_0}{d}\right)$$

On the surface of the lower plate,

$$\boldsymbol{e}_n = \boldsymbol{e}_z, \quad \rho_s = -\frac{\varepsilon U_0}{d}$$

On the surface of the upper plate,

$$\boldsymbol{e}_n = -\boldsymbol{e}_z, \quad \rho_s = \frac{\varepsilon U_0}{d}$$

This agrees with the fact that electric field lines in an electrostatic field originate from positive charges and terminate in negative charges.

Example 3-2 A cylindrical capacitor consists of an inner conductor of radius a and an outer conductor whose inner radius is b. The space between the conductors is filled with a dielectric of permittivity ε, and the length of the capacitor is L. The outer conductor is grounded, and the inner conductor is maintained at potential U_0. Determine ① the potential distribution between the two conductors, and ② the capacitance of this capacitor.

Solution: This is the same problem as Example 2-17, which is solved by applying Gauss's law. Here we solve it by solving the one-dimensional Laplace's equation under the cylindrical coordinate system.

(1) Due to cylindrical symmetry, φ has no variation along the ϕ-and z-directions (assuming no fringing effect). Laplace's equation (3.9) is then simplified to

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{d\varphi}{d\rho}\right) = 0 \tag{3.16}$$

Integration of (3.16) with respect to ρ gives

$$\frac{d\varphi}{d\rho} = \frac{C_1}{\rho}$$

where C_1 is an unknown constant coefficient. Integrating again, we obtain

$$\varphi = C_1 \ln\rho + C_2 \tag{3.17}$$

To determine the two unknown coefficients C_1 and C_2, two boundary conditions are used:

$$\text{At } \rho = b, \quad \varphi = 0 \tag{3.18a}$$

$$\text{At } \rho = a, \quad \varphi = U_0 \tag{3.18b}$$

Substitution of (3.18a) and (3.18b) into (3.17) yields two equations, from which the two unknowns are solved to be $C_1 = U_0/\ln(a/b)$ and $C_2 = -U_0 \ln(b)/\ln(a/b)$. Hence the potential distribution between the conductors is

$$\varphi = \frac{U_0}{\ln\left(\frac{a}{b}\right)} \ln\left(\frac{\rho}{b}\right) \tag{3.19}$$

(2) In order to find the capacitance, we first find the distribution of $\boldsymbol{E}$ within the capacitor. From(3.3) and(3.19) we have

$$\boldsymbol{E}(\rho) = -\boldsymbol{e}_\rho \frac{\mathrm{d}\varphi}{\mathrm{d}\rho} = -\boldsymbol{e}_\rho \frac{U_0}{\ln\left(\frac{a}{b}\right)} \frac{1}{\rho} \tag{3.20}$$

At the surface of the inner conductor($\rho = a$), we have

$$E_n(a) = \boldsymbol{e}_n \cdot \boldsymbol{E}(a) = \boldsymbol{e}_\rho \cdot (-\boldsymbol{e}_\rho) \frac{U_0}{\ln\left(\frac{a}{b}\right)} \frac{1}{a} = \frac{U_0}{\ln\left(\frac{b}{a}\right)} \frac{1}{a}$$

which is a constant. The surface charge densities at the conducting plates are obtained by using the boundary condition, i. e. ,

$$\rho_s = \varepsilon E_n = \frac{\varepsilon U_0}{\ln\left(\frac{b}{a}\right)} \frac{1}{a}$$

The total charge on the inner conductor is

$$Q = \int_S \rho_s \mathrm{d}s = 2\pi a L \rho_s = \frac{2\pi\varepsilon L U_0}{\ln\left(\frac{b}{a}\right)} \tag{3.21}$$

We can verify easily that the charge carried by the outer conductor is $-Q$. Therefore, the capacitance is calculated as

$$C = \frac{Q}{U_0} = \frac{2\pi\varepsilon L}{\ln\left(\frac{b}{a}\right)}$$

which is the same as the result of Example 2-17.

Example 3-3 A spherical capacitor consists of an inner conducting sphere of radius a and an outer conductor with inner radius b. The space in between is filled with a dielectric of permittivity ε. The outer conductor is grounded, and the inner conductor is maintained at a potential U_0. Determine ① the potential distribution between the two conductors, and ② the capacitance of this capacitor.

Solution: This is essentially the same problem as Example 2-18. Here we solve it based on the Laplace's equation in spherical coordinates.

(1) Due to symmetry, φ has no variation along the ϕ- and θ- directions. Hence φ between the two conductors satisfies the one-dimensional Laplace's equation

$$\frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\varphi}{\mathrm{d}r}\right) = 0 \tag{3.22}$$

Integration of(3.22) with respect to r gives

$$\frac{d\varphi}{dr}=\frac{C_1}{r^2}$$

where C_1 is an unknown constant coefficient. Integrating again, we obtain

$$\varphi=-\frac{C_1}{r}+C_2 \tag{3.23}$$

To determine the two unknown coefficients C_1 and C_2, two boundary conditions are used:

$$\text{At } r=b,\quad \varphi=0 \tag{3.24a}$$

$$\text{At } r=a,\quad \varphi=U_0 \tag{3.24b}$$

which leads to the solution of φ as

$$\varphi=\frac{U_0}{\frac{1}{a}-\frac{1}{b}}\left(\frac{1}{r}-\frac{1}{b}\right) \tag{3.25}$$

(2) From(3.3) and(3.25) we have

$$\boldsymbol{E}=-\boldsymbol{e}_{\mathrm{r}}\frac{d\varphi}{dr}=\boldsymbol{e}_{\mathrm{r}}\frac{U_0}{\frac{1}{a}-\frac{1}{b}}\left(\frac{1}{r^2}\right) \tag{3.26}$$

At the surface of the inner conductor($r=a$), we have

$$E_{\mathrm{n}}(a)=\boldsymbol{e}_{\mathrm{n}}\cdot\boldsymbol{E}(a)=\boldsymbol{e}_{\mathrm{r}}\cdot\boldsymbol{e}_{\mathrm{r}}\frac{U_0}{\frac{1}{a}-\frac{1}{b}}\left(\frac{1}{a^2}\right)=\frac{U_0}{\frac{1}{a}-\frac{1}{b}}\left(\frac{1}{a^2}\right)$$

The surface charge density at the inner conductor is obtained by using the boundary condition, i.e.,

$$\rho_{\mathrm{s}}=\varepsilon E_{\mathrm{n}}=\frac{\varepsilon U_0}{\frac{1}{a}-\frac{1}{b}}\left(\frac{1}{a^2}\right)$$

The total charge on the inner conductor is

$$Q=\int_S\rho_{\mathrm{s}}ds=4\pi a^2\rho_{\mathrm{s}}=\frac{4\pi\varepsilon U_0}{\frac{1}{a}-\frac{1}{b}} \tag{3.27}$$

We can verify easily that the charge carried by the outer conductor is $-Q$. Therefore, the capacitance is calculated as

$$C=\frac{Q}{U_0}=\frac{4\pi\varepsilon}{\frac{1}{a}-\frac{1}{b}}$$

which is the same as the result of Example 2-18.

Example 3-4 Determine the $\boldsymbol{E}$ field caused by a uniform charge distribution in a sphere with a volume density $\rho_{\mathrm{v}}=\rho_0$ for $0\leqslant r\leqslant a$ and $\rho_{\mathrm{v}}=0$ for $r>a$.

Solution: This is the same problem as Example 2-7, which is solved by applying Gauss's law. Here we solve it by direct solving the one-dimensional Poisson's and Laplace's equations. By the spherical symmetry, there are no variations in θ- and ϕ- direction. Therefore, the fields including $\boldsymbol{E}$ and φ are functions of the r coordinates only. Both Poisson's and Laplace's equations are reduced to one-dimensional.

(1) For region $0 \leqslant r \leqslant a$, $\rho_v = \rho_0$. The potential must satisfy 1-D Poisson's equation

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\varphi}{\mathrm{d}r}\right) = -\frac{\rho_0}{\varepsilon_0}$$

Integration of the above equation gives

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = -\frac{\rho_0}{3\varepsilon_0}r + \frac{C_1}{r^2} \tag{3.28}$$

Therefore, the electric field intensity inside the region is

$$\boldsymbol{E} = -\nabla\varphi = -\boldsymbol{e}_r\left(\frac{\mathrm{d}\varphi}{\mathrm{d}r}\right) = \boldsymbol{e}_r\frac{\rho_0}{3\varepsilon_0}r \quad (0 \leqslant r \leqslant a) \tag{3.29}$$

Here, we have used the fact that C_1 in (3.28) must be zero because otherwise, $\boldsymbol{E}$ will become infinite at $r=0$.

(2) For region $r>a$, $\rho_v = 0$. The potential must satisfy 1-D Laplace's equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\mathrm{d}\varphi}{\mathrm{d}r}\right) = 0 \tag{3.30}$$

Integrating of the above equation gives

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \frac{C_2}{r^2} \tag{3.31}$$

and therefore,

$$\boldsymbol{E} = -\nabla\varphi = -\boldsymbol{e}_r\frac{\mathrm{d}\varphi}{\mathrm{d}r} = -\boldsymbol{e}_r\frac{C_2}{r^2} \quad (r > a) \tag{3.32}$$

The integration constant C_2 can be found by equating $\boldsymbol{E}$ at $r = a$, which is the boundary condition of normal continuity of $\boldsymbol{D}$ vector (the permittivity is the same ε_0 inside and outside the source region). Therefore, from (3.29) and (3.32), we have

$$\frac{\rho_0}{3\varepsilon_0}a = -\frac{C_2}{a^2}$$

which gives

$$C_2 = -\frac{\rho_0 a^3}{3\varepsilon_0} \tag{3.33}$$

Substitution of (3.33) into (3.32) gives

$$\boldsymbol{E} = \boldsymbol{e}_r\frac{\rho_0 a^3}{3\varepsilon_0 r^2} \quad (r > a) \tag{3.34}$$

which is the same as the results obtained in Example 2-7. We can continue to find the potential distribution as a function of r. For the region $0 \leqslant r \leqslant a$, integrating (3.28) in which C_1 is already determined to be zero, we have

$$\varphi = -\frac{\rho_0 r^2}{6\varepsilon_0} + C_1' \quad (0 \leqslant r \leqslant a) \tag{3.35}$$

where C'_1 is a new integration constant that will be determined later. For the region $r > a$, substituting (3.33) into (3.31) and integrating both sides of the resulted equation, we obtain

$$\varphi = \frac{\rho_0 a^3}{3\varepsilon_0 r} \quad (r > a) \tag{3.36}$$

Here, we do not include an additional unknown constant in the integration result because the potential φ is zero at infinity ($r \to \infty$). The only unknown left is C_1' in (3.35), which can be determined by the continuity condition of φ across the boundary. Let φ in (3.35) and (3.36) be equal at the boundary $r = a$, we have

$$-\frac{\rho_0 a^2}{6\varepsilon_0} + C_1' = \frac{\rho_0 a^2}{3\varepsilon_0}$$

then

$$C_1' = \frac{\rho_0 a^2}{2\varepsilon_0} \tag{3.37}$$

Substitute (3.37) into (3.35), we have

$$\varphi = \frac{\rho_0}{3\varepsilon_0}\left(\frac{3a^2}{2} - \frac{r^2}{2}\right) \quad (0 \leqslant r \leqslant a) \tag{3.38}$$

3.3 Uniqueness of Electrostatic Solutions (静电场解的唯一性)

In the examples in the last section, we obtained the solutions by direct integration. However, direct integration can be used only if Poisson's (or Laplace's) equation is reduced to one dimensional due to the symmetry. In more complicated situations involving two- or three-dimensional partial differential equations, the solution usually cannot be obtained by direct integration. Nevertheless, in some special cases, analytical or semi-analytical solutions can still be obtained by using special methods such as the method of images and the method of separation of variables that will be introduced later in this chapter. These two methods are both based on the important **uniqueness theorem** (唯一性定理).

The uniqueness theorem states that the solution of Poisson's (or Laplace's) equation satisfying the given boundary conditions is a unique solution.① This means that, no matter what method we use to obtain a solution of the Poisson's (or Laplace's), it must be the correct solution as long as the boundary conditions are satisfied.

To prove the uniqueness theorem, we take an arbitrary volume V bounded by a closed surface S_o which may be a surface at infinity. Inside the closed surface S_o, the volume V may also be bounded by some interior surfaces $S_1, S_2, \cdots, S_N$ as depicted in Figure 3-2. Now assume

① 静电场解的唯一性定理可以表述为：满足给定边界条件的泊松方程或拉普拉斯方程的解是唯一存在的。

that there are two solutions, φ_1 and φ_2, to the same Poisson's equation in V, i. e. ,

$$\nabla^2\varphi_1 = -\frac{\rho_v}{\varepsilon} \tag{3.39a}$$

$$\nabla^2\varphi_2 = -\frac{\rho_v}{\varepsilon} \tag{3.39b}$$

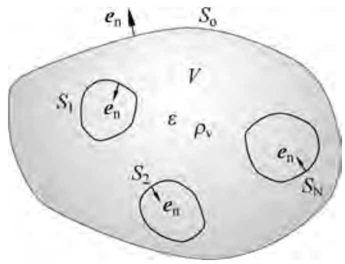

Figure 3-2 A region V bounded by an external surface S_o and possible internal surfaces $S_1, S_2, \cdots, S_N$

where ρ_v is the charge density within the volume V. Then we only need to prove that the difference between φ_1 and φ_2 in the volume V must be zero if φ_1 and φ_2 satisfy the same boundary conditions on $S_1, S_2, \cdots, S_N$ and S_o. To do that, we define a difference potential:

$$\varphi_d = \varphi_1 - \varphi_2 \tag{3.40}$$

From(3. 39a) and (3. 39b), it is obvious that φ_d must satisfy Laplace's equation in the volume V

$$\nabla^2\varphi_d = 0 \tag{3.41}$$

Utilizing the vector identity(1. 148), in which let $\psi=\varphi_d$ and $\boldsymbol{A}=\nabla\varphi_d$, we have

$$\nabla\cdot(\varphi_d\nabla\varphi_d) = \varphi_d\nabla^2\varphi_d + |\nabla\varphi_d|^2 \tag{3.42}$$

From(3. 41), the first term on the right side of(3. 42) vanishes. Integrating both sides of(3. 42) over the volume V and applying the divergence theorem to the left side, we have

$$\oint_S(\varphi_d\nabla\varphi_d)\cdot\boldsymbol{e}_n\,ds = \int_V|\nabla\varphi_d|^2dv \tag{3.43}$$

where $\boldsymbol{e}_n$ denotes the unit normal outward from V, and the surface S consists of S_o as well as S_1, $S_2, \cdots, S_N$. Noticing that $\nabla\varphi_d\cdot\boldsymbol{e}_n=\partial\varphi_d/\partial n$, (3. 43) can be rewritten as

$$\oint_S\varphi_d\frac{\partial\varphi_d}{\partial n}ds = \int_V|\nabla\varphi_d|^2dv \tag{3.44}$$

Now, we only need to show that(3. 44) implies φ_d must be zero if φ_1 and φ_2 satisfies the same boundary conditions. The boundary conditions can take different forms depending on the specific electrostatic problems. Typical forms of the boundary conditions include but not limited to the following.

(1) The potential φ is specified on some or all the boundaries. Then $\varphi_1=\varphi_2$ on these boundaries, and therefore, φ_d on these boundaries is identically zero;

(2) $\partial\varphi/\partial n$ is specified on some or all the boundaries (which is equivalent to specified surface charge densities if these boundaries are conductor-dielectric interfaces). Then $\partial\varphi_1/\partial n=\partial\varphi_2/\partial n$ on these boundaries, and therefore, $\partial\varphi_d/\partial n$ on these boundaries is identically zero;

(3) If S_o (or partial S_o) is at infinity, it can be considered as the surface (or partial surface) of a sphere centered at origin with a radius r approaching infinity. As r increases, both φ_1 and φ_2 decrease as $1/r$ (if the charge distribution is within a bounded region, which is true for most practical problems). Hence φ_d decrease as $1/r$ and $\nabla\varphi_d$ decreases as $1/r^2$, making the integrand $\varphi_d(\partial\varphi_d/\partial n)$ decreases as $1/r^3$. As the surface area of S_o (or partial S_o) increases as r^2, the surface integral of $\varphi_d(\partial\varphi_d/\partial n)$ on S_o (or partial S_o) decreases as $1/r$ and approaches zero at infinity.

All the above cases lead to the conclusion that the surface integral on the left side of (3.44) is zero, and as a result, the volume integral on the right side of (3.44) must also be zero, i.e.,

$$\int_V |\nabla\varphi_d|^2 dv = 0 \tag{3.45}$$

Since the integrand $|\nabla\varphi_d|^2$ is nonnegative everywhere, (3.45) can be satisfied only if $|\nabla\varphi_d|^2$ is zero everywhere inside the volume V. The gradient of φ_d is everywhere zero, meaning that φ_d is constant at all points in V. Therefore, φ_1 can be different from φ_2 by only a constant. However, as we know, a constant difference in potential distribution does not make any difference in electric fields.① And the constant difference can be eliminated by selecting the same reference zero potential point in the solution of φ_1 and φ_2, in which case $\varphi_1 = \varphi_2$. This proves that there is only one possible solution.②

3.4 Method of Images(镜像法)

There is a class of electrostatic problems that can be simplified by replacing bounding surfaces by appropriate image charges. This method is called the **method of images**(镜像法).③

3.4.1 Image with Respect to Planes(平面镜像)

To illustrate the method of images, we consider the problem of finding electrostatic field produced by a point charge in front of an infinitely large grounded conducting plane. As shown in Figure 3-3(a), a positive point charge Q is located at a distance d above conducting plane. Here, the objective is to solve for the potential everywhere above the conducting plane ($z>0$). It can be formulated as the boundary-value problem of solving Poisson's equation:

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = -\frac{Q\delta(\boldsymbol{r} - \boldsymbol{d})}{\varepsilon_0} \quad (z > 0) \tag{3.46}$$

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① 电位分布加上任意一个常数,其空间变化率不会发生变化,由电位分布确定的电场强度也不会变化。

② 唯一性定理的重要意义在于:①给出了静态场边值问题具有唯一解的条件;②为静态场边值问题的各种求解方法提供了理论依据;③为求解结果的正确性提供了判据。3.4 节和 3.5 节介绍基于唯一性定理的两种特殊且很重要的静电场边值问题求解方法。

③ 镜像法的基本思想是引入位于边界外虚设的较简单的镜像电荷分布来等效替代该边界上未知的较为复杂的电荷分布,从而将原本带有复杂电荷分布的边界值问题转换成无限大单一均匀媒质空间已知电荷分布求电场的问题,简化分析计算过程。镜像法的理论依据是唯一性定理。

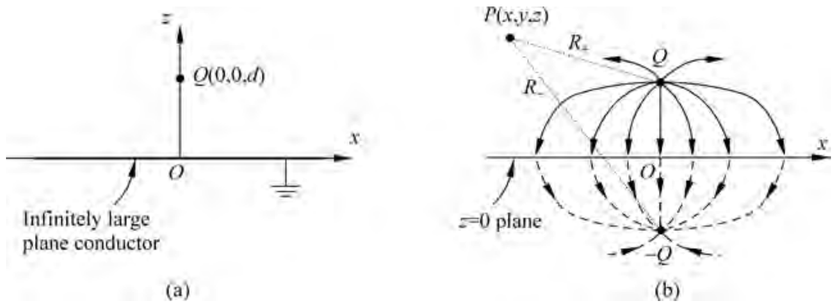

Figure 3-3 Point charge in front of a grounded plane conductor

subject to the boundary conditions

$$\varphi(x,y,0)=0 \tag{3.47}$$

and

$$\varphi(x,y,z)\to 0,\quad \text{as}\quad x\to\pm\infty,y\to\pm\infty \quad \text{or} \quad z\to+\infty \tag{3.48}$$

In(3.46), the volume charge density of the point charge Q is represented by $Q\delta(\boldsymbol{r}-\boldsymbol{d})$, where $\boldsymbol{d}=\boldsymbol{e}_z d$ is the position vector of the location of the point charge.

Obviously, φ in this problem is a field depending on all the three coordinates x, y and z. Therefore, we cannot construct its solution by direct integration of the equation(3.46).

From the physical point of view, the positive charge Q at $z=d$ induces negative charges on the surface of the conducting plane, resulting in a surface charge density ρ_s. Hence the potential to solve can be written as

$$\varphi(x,y,z)=\frac{Q}{4\pi\varepsilon_0\sqrt{x^2+y^2+(z-d)^2}}+\frac{1}{4\pi\varepsilon_0}\int_S\frac{\rho_s(x',y')}{\sqrt{(x-x')^2+(y-y')^2+z^2}}\mathrm{d}x'\mathrm{d}y'$$

where S is the surface of the plane conductor. Unfortunately, the induced surface charge distribution ρ_s is unknown. Moreover, it is quite difficult to evaluate the surface integral in the above expression even if ρ_s is found. However, with the method of images, this problem can be easily solved, which is demonstrated as follows.

As has been pointed out, it is the unknown ρ_s on the surface of the conducting plane that causes the trouble in solving this problem. In the method of images, we remove the conducting plane together with the induced charges and replace them with an image point charge $-Q$ at $z=-d$ as shown in Figure 3-3(b). Then, the potential at a point $P(x,y,z)$ in the $z>0$ region can be easily found as

$$\varphi(x,y,z)=\frac{Q}{4\pi\varepsilon}\left(\frac{1}{R_+}-\frac{1}{R_-}\right) \tag{3.49}$$

where R_+ and R_- are respectively the distances from $+Q$ and $-Q$ to the field point(x,y,z), i.e.,

$$R_+=\sqrt{x^2+y^2+(z-d)^2} \quad \text{and} \quad R_-=\sqrt{x^2+y^2+(z+d)^2}$$

Now we need to verify that the potential expression of(3.49) is exactly the solution of the electrostatic problem of Figure 3-3(a) in the $z>0$ region.

In the $z>0$ region, the medium and source distribution in the problem of Figure 3-3(a) are the same as those in Figure 3-3(b). Therefore, it is apparent that(3.49) satisfies the governing equation(3.46). It is also obvious that(3.49) satisfies the boundary conditions(3.47) and (3.48). Therefore, (3.49) gives a potential field that satisfies the same equation and the same boundary conditions in the $z>0$ region as specified in the problem of Figure 3-3(a). According to the uniqueness theorem, (3.49) must be the solution of the problem of Figure 3-3(a) in the $z>0$ region. ①

With the solution of potential φ, electric field intensity $\boldsymbol{E}$ in the $z>0$ region can be found by taking the negative gradient of φ. A few of the field lines are shown in Figure 3-3(b). The induced surface charge distribution ρ_s can be found by taking the negative directional derivative of φ along the normal direction on the conductor surface. Notice that, in the $z<0$ region, the potential field solution of Figure 3-3(b) is not the same as that of Figure 3-3(a). Apparently, the field is zero in the $z<0$ region in Figure 3-3(a). But in Figure 3-3(b), the field is nonzero as indicated by the dashed electric field lines.

Now we see that the method of images significantly simplifies the solution of this electrostatic problem of Figure 3-3(a). This is achieved by introducing a simple **image charge** (**镜像电荷**) that is equivalent to the unknown charge distribution on the boundary. It is important to realize that introduction of the image charge should not change anything within the region in which the field is to be determined($z>0$ in this problem). In other words, the image charges must be located outside the region of interest($z<0$ in this problem). Outside the region of interest, (3.49) is still the solution of the problem in Figure 3-3(b), but not the solution of the problem in Figure 3-3(a) anymore. As a matter of fact, both φ and $\boldsymbol{E}$ are zero in the $z<0$ region in Figure 3-3(a).

A similar problem is the electric field due to a line charge ρ_l above an infinite conducting plane, which can be found from ρ_l and its image $-\rho_l$(with the conducting plane removed).

Example 3-5 As is shown in Figure 3-4(a), a positive point charge Q is located in the first quadrant ($x>0$, $y>0$) that is bounded by two orthogonal conducting planes that are grounded. The point charge is d_1 and d_2 from the two planes. Determine the potential distribution within the first quadrant.

Solution: To solve this problem by using the method of images, we need to find the image charges that can replace the effect of the two conducting half-planes. The image charges should be outside the first quadrant. After the conducting half-planes are replaced by the image charges, the potential at the locations of the half-planes should remain to be zero. To make the potential of the horizontal half-plane zero, we can first add an image charge $-Q$ in the fourth quadrant. Then to make the potential of the vertical half-plane zero, we add an image charge $-Q$ in the

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① 图 3-3(a)所示的问题中,点电荷 Q 在无限大接地导体平面感应出的电荷分布可以等效替换为距离导体平面相同距离的另一侧的镜像电荷 $-Q$,即图 3-3(b)所示的问题。能够做上述等效替换是基于唯一性定理。具体而言,图 3-3(a)与图 3-3(b)所示的两个问题中的电位在 $y>0$ 的区域内满足同样的泊松方程和同样的边界条件,因此在该区域中电位的解必然是相同的。通过这个例子也可以看到应用镜像原理的两个原则:镜像电荷必须位于所求解的场区域以外;镜像电荷的个数、位置及电荷量的大小由满足所求解的场区域的边界条件确定。

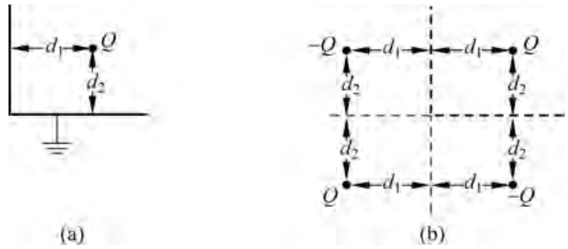

Figure 3-4 Point charge in front of two perpendicular conducting half planes

second quadrant. However, the image charge in the fourth quadrant produces a non-zero potential on the vertical half-plane, and the one in the second quadrant produces a non-zero potential on the horizontal half-plane. To balance out the non-zero potentials, we can introduce a third image charge $+Q$ in the third quadrant. With the three image charges as shown in Figure 3-4(b), it can be easily verified that the zero-potential boundary conditions on both half-planes are satisfied. According to the uniqueness theorem, the effect of the two conducting half-planes can be replaced by the image charges. The potential and electric field distribution in the first quadrant in Figure 3-4(b) is the same as that in Figure 3-4(a). Therefore, we have

$$\varphi(x,y,z) = \frac{Q}{4\pi\varepsilon_0\sqrt{(x-d_1)^2+(y-d_2)^2+z^2}} - \frac{Q}{4\pi\varepsilon_0\sqrt{(x+d_1)^2+(y-d_2)^2+z^2}}$$
$$- \frac{Q}{4\pi\varepsilon_0\sqrt{(x-d_1)^2+(y+d_2)^2+z^2}} + \frac{Q}{4\pi\varepsilon_0\sqrt{(x+d_1)^2+(y+d_2)^2+z^2}}$$

The electric field intensity in the first quadrant and the surface charge density induced on the two half-planes can also be found from the system of four charges.

As an extension of Example 3-5, if the angle α made by the two intersecting half planes are other than 90°, the method of image may still be used to find the solution of the fields due to a point charge. The number of image charges needed depends on the angle. Specifically, if the angle $\alpha = 180°/n$ with n to be a positive integer, $(2n-1)$ image charges is needed to replace the conducting half planes. Otherwise, infinite number of image charges are required, in which case, an approximate solution can be found by ignoring those too far away from the region of interests.

3.4.2 Image with Respect to Spheres(球面镜像)

Here we consider the electrostatic problem of a point charge in front of a spherical conductor. As is shown in Figure 3-5(a), a positive point charge Q is located at a distance d from the center of a grounded conducting sphere of radius $a(a<d)$. The problem is to find the φ and $\boldsymbol{E}$ field distributions outside the sphere. Apparently, the difficulty of this problem lies in the unknown induced charge distribution on the surface of the conducting sphere. This difficulty can be circumvented if an image point charge Q_i can be found to replace the effect of the sphere. If this image charge Q_i exists, we must have the following:

(1) Q_i must be a negative charge inside the sphere and on the line OQ due to geometrical symmetry.

(2) After the conducting sphere is replaced by the image charge Q_i, the boundary condition on the spherical surface must remain unchanged. In other words, the potential at $r=a$ should be zero.

Now let's prove such image charge Q_i does exist as is illustrated in Figure 3-5(b). Q_i cannot be equal to $-Q$, because $-Q$ and the original Q do not make the spherical surface $r=a$ the zero-potential surface as required. Therefore, Q_i is an unknown. Another unknown is the distance between Q_i and the origin O, denoted by d_i. To find the solution of d_i and Q_i, we first write down the potential caused by Q and Q_i at a point M as

$$\varphi_M = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{R} + \frac{Q_i}{R'}\right)$$

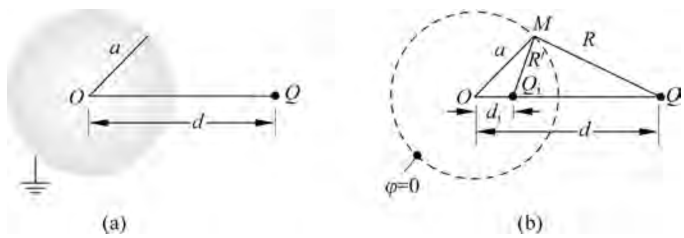

Figure 3-5 Point charge in front of a grounded sphere

where R and R' are respectively the distance from Q and Q_i to the point M. The boundary condition is $\varphi_M=0$ for any point M on the $r=a$ surface, which requires

$$\frac{R'}{R} = -\frac{Q_i}{Q} = \text{constant} \qquad (3.50)$$

while the point M travels on the spherical surface. This condition can be satisfied by simply selecting d_i so that triangles $\angle OMQ_i$ and $\angle OQM$ are similar. Notice that the two triangles have one common angle $\angle MOQ_i = \angle QOM$, and the edges $\overline{OM}=a$, $\overline{OQ}=d$ are constant lengths. If we select $\overline{OQ_i}=d_i$ so that

$$\frac{\overline{OQ_i}}{\overline{OM}} = \frac{\overline{OM}}{\overline{OP}}$$

then the two triangles become similar, and we have

$$\frac{d_i}{a} = \frac{a}{d} = \frac{R'}{R} \qquad (3.51)$$

from which we immediately find that

$$d_i = \frac{a^2}{d} \qquad (3.52)$$

From (3.51), the constant ratio in (3.50) must be a/d, and hence

$$Q_i = -\frac{a}{d}Q \qquad (3.53)$$

Now we see that, with d_i and Q_i given by (3.52) and (3.53), the potential field in Figure 3-5(b) satisfies the same boundary condition as in Figure 3-5(a). Therefore, Q_i must be the image charge of Q with respect to the spherical surface $r=a$. The φ and $\boldsymbol{E}$ of all points external to the grounded sphere can now be calculated as if they are produced by the point charges Q and Q_i. Specifically, as shown in Figure 3-6, the electric potential φ at an arbitrary point $P(r,\theta)$ is

$$\varphi(r,\theta) = \frac{1}{4\pi\varepsilon}\left(\frac{Q}{R} - \frac{a}{d}\frac{Q}{R'}\right) \tag{3.54}$$

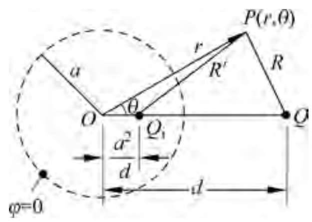

Figure 3-6 Image method solution of the problem in Figure 3-5

By the law of cosines,

$$R = \sqrt{r^2 + d^2 - 2rd\cos\theta} \tag{3.55}$$

and

$$R' = \sqrt{r^2 + (a^2/d)^2 - 2r(a^2/d)\cos\theta} \tag{3.56}$$

Substitute (3.55) and (3.56) into (3.54), then the r-component of the $\boldsymbol{E}$ field can be calculated as

$$\begin{aligned} E_r(r,\theta) &= -\frac{\partial\varphi(r,\theta)}{\partial r} \\ &= \frac{Q}{4\pi\varepsilon_0}\left\{\frac{r - d\cos\theta}{(r^2 + d^2 - 2rd\cos\theta)^{3/2}} - \frac{a[r-(a^2/d)\cos\theta]}{d[r^2 + (a^2/d)^2 - 2r(a^2/d)\cos\theta]^{3/2}}\right\} \end{aligned} \tag{3.57}$$

With (3.57), we can find the induced surface charge on the sphere by letting $r=a$, and after some mathematical manipulation, we have

$$\rho_s = \varepsilon_0 E_r(a,\theta) = -\frac{Q(d^2 - a^2)}{4\pi a(a^2 + d^2 - 2ad\cos\theta)^{3/2}} \tag{3.58}$$

(3.58) tells us that the induced surface charge is negative and that its magnitude is maximum at $\theta=0$ and minimum at $\theta=\pi$, as expected.

The total charge induced on the sphere is an integration of ρ_s given by (3.58) over the surface of the sphere, i.e.,

$$Q_{\text{induced}} = \oint \rho_s \mathrm{d}s = \int_0^{2\pi}\int_0^{\pi} \rho_s a^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi = -\frac{a}{d}Q = Q_i \tag{3.59}$$

Note that the total induced charge is exactly equal to the image charge Q_i.

Example 3-6 A point charge Q is located outside an isolated conducting sphere with a distance d from the center of the sphere. As is illustrated in Figure 3-7(a), the conducting sphere has a radius a. Determine the image of the charge Q with respect to the surface of the conducting sphere.

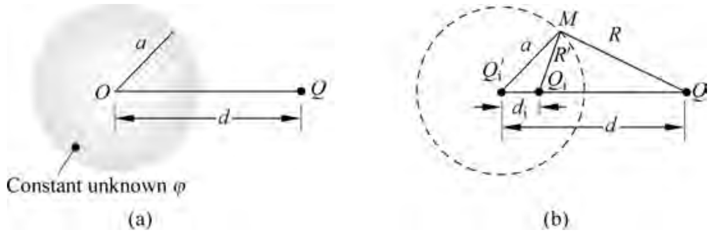

Figure 3-7 Point charge in front of an isolated conducting sphere

Solution: Different from the problem of Figure 3-5, the sphere is isolated, which means the potential on the sphere surface is not zero. Nevertheless, the sphere surface is still equipotential, which can be realized by the image charge Q_i and its location given by (3.53) and (3.52). However, Q_i and Q together make the sphere surface a zero-potential surface, whereas in this example, the potential on the sphere surface is a non-zero constant. This constant potential is unknown, but we know the isolated sphere is neutral, which means the total image charges must also be zero (why?). Therefore, as is shown in Figure 3-7(b), we can introduce an additional image charge

$$Q_i' = -Q_i = \frac{a}{d}Q \tag{3.60}$$

at the sphere center to make the net image charge zero. Q_i' must be located at the sphere center so that the potential on the $r=a$ surface remains constant. Then, the original problem can be solved as a problem with three point charges: Q_i' at $r=0$, Q_i at $r=a^2/d$, and the original Q at $r=d$. ①

3.4.3 Image in Cylinders (圆柱面镜像)

Consider a line charge ρ_l outside of a parallel, conducting, circular cylinder with radius a as shown in Figure 3-8(a). The distance between the line charge and the axis of the cylinder is d. The problem is to find the field distributions outside the cylinder. Again, the difficulty of this problem lies in the unknown induced charge distribution on the surface of the conducting

① 例 3-6 推断孤立导体球外点电荷 Q 关于球面的镜像电荷有两个 Q_i 和 Q_i'。其依据是 Q、Q_i 和 Q_i' 三个电荷共同产生的电位在球面上是常数，且球面包围的总电荷必须为零。然而，由于原问题中导体球表面的电位或者电位的法向导数均未知，上述依据实际上不足以说明这两个镜像电荷代替导体形成的电位分布与原问题中电位分布满足同样的边界条件。要严格地证明例题中镜像法得到的解就是真实的解，需要回到唯一性定理的证明：假设 φ_d 为镜像法得到的电位与真实电位之差，则 φ_d 在导体球外满足式(3.44)。由于镜像法得到的解与真实解在导体表面都为常数，φ_d 在导体表面必然也是常数，因此式(3.44)的左边可以写 $\varphi_d\oint_S(\partial\varphi_d/\partial n)\,ds = -\varphi_d\oint_S E_{nd}\,ds$，其中 E_{nd} 为镜像法得到的电场与真实电场在导体球表面的法向分量之差。根据高斯定律，镜像法得到的电场与真实电场的法向分量在导体球表面的通量都等于零（球面包含的总电荷量均为零）。因此，$\oint_S E_{nd}\,ds = 0$，故式(3.44)左边等于零，式(3.44)的右边 $\int_V |\nabla\varphi_d|^2 dv = 0$，从而证明了 φ_d 在导体球外的整个区域均为零。

cylinder, which can be solved by using the method of image. We first recognize the following:

(1) The image must be a parallel line charge (denoted by ρ_i) inside the cylinder, and it must lie somewhere along OP, due to the symmetry of the geometry.

(2) After the conducting cylinder is replaced by the image charge, the boundary condition on the cylindrical surface remains unchanged. Particularly, the potential at $\rho = a$ should be constant.

Let the distance between the image charge and the axis be d_i as shown in Figure 3-8(b). Then we need to determine the two unknowns, ρ_i and d_i.

Recall that, in Example 2-11, the equipotential surfaces of the field produced by two parallel line charges, ρ_l and $-\rho_l$, are circular cylindrical surfaces. If one of the equipotential surfaces coincides with the surface of the conducting cylinder in Figure 3-8(a), then according to the uniqueness theorem, the conducting cylinder can be replaced by the line charge $-\rho_l$ in Example 2-11. Therefore, we infer that the image of the line charge ρ_l in Figure 3-8(a) to be[①]

$$\rho_i = -\rho_l \tag{3.61}$$

To find d_i, we first write down the expression of the potential due to the line charges ρ_i and ρ_l. According to Example 2-11, at any point M on the cylindrical surface $\rho = a$, we have

$$\varphi_M = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{\rho'}{\rho} \tag{3.62}$$

where ρ and ρ' are the distances from the point M to the line charges ρ_l and ρ_i respectively as is shown in Figure 3-8(b). Obviously, the boundary condition requires that ρ'/ρ maintain constant when the point M travels on the cylindrical surface. This condition can be satisfied by simply selecting a d_i value so that triangles $\angle OMP_i$ and $\angle OPM$ are similar. Note that these two triangles have one common angle $\angle MOP_i = \angle POM$. Hence the two triangles can be made similar by letting

$$\frac{\overline{OP_i}}{\overline{OM}} = \frac{\overline{OM}}{\overline{OP}}$$

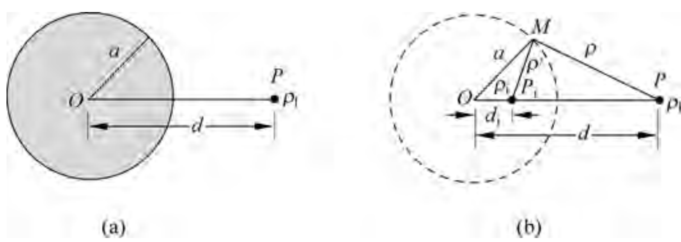

Figure 3-8 Line charge in front of a parallel conducting circular cylinder

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① 这里假设无限长导体柱单位长度所带电荷为 $-\rho_l$。需要指出，由于导体柱无限长，导体外线电荷 ρ_l 也分布到无穷远处，该问题并没有设定导体柱是否接地，也没有给定导体柱的带电量。如果认为该导体柱接地，那么无穷远处就是参考零电位点，由例题 2-11 可以推断导体柱单位长度所带电荷必须为 $-\rho_l$。如果导体柱单位长度所带电荷不是 $-\rho_l$，就不能选取无穷远处为参考零电位点，而该问题的镜像电荷应该再加上一个位于导体柱轴线上的线电荷，其密度为导体柱单位长度实际所带电荷与 $-\rho_l$ 之差。而根据例题 2-10，该位于导体柱轴线上的镜像电荷的电位在无穷远处为无穷大。

Since $\overline{OM}=a$, $\overline{OP}=d$ and $\overline{OP_i}=d_i$, the above relation is satisfied by letting

$$d_i = \frac{a^2}{d} \tag{3.63}$$

And as a result,

$$\frac{\rho'}{\rho} = \frac{d_i}{a} = \frac{a}{d} = \text{constant} \tag{3.64}$$

for any point M on the cylindrical surface. By substituting (3.64) into (3.62), the constant potential on the cylindrical surface is

$$\varphi_M = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a}{d} \tag{3.65}$$

Now, it is verified that the line charge $\rho_i = -\rho_l$ is the image of the original line charge ρ_l with respect to the cylindrical conducting surface $\rho = a$, and the fields at any point outside the surface can be determined equivalently by ρ_l and ρ_i.

The above discussion demonstrates that a cylindrical conductor with surface charges induced by an external line charge can be replaced by an internal line charge. This conclusion is useful in determining the capacitance of two wire transmission lines as demonstrated in the following example.

Example 3-7 Two-wire transmission line: as shown in Figure 3-9(a), two infinitely long conducting wires of radius a are parallel to each other with a distance D between the axes. Determine the capacitance per unit length between the two wires.

Solution: As shown in Figure 3-9(b), the two conducting wires can be replaced by a pair of line charges $+\rho_l$ and $-\rho_l$, as long as the potential generated by the two line charges is constant on each of the cylindrical surfaces. Referring to the method of image used in the problem of Figure 3-8, the separation between the image charge within one cylinder and the axis of the other cylinder should be $d = D - d_i$. Using (3.63), we have

$$d = D - d_i = D - \frac{a^2}{d}$$

from which we obtain

$$d = \frac{1}{2}\left(D + \sqrt{D^2 - 4a^2}\right) \tag{3.66}$$

The potential difference between the two wires is that between any two points on the respective wires. Using (3.65), the potential on the cylinder surface surrounding positive line charge $+\rho_l$ is

$$\varphi_+ = -\frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a}{d}$$

The potential on the cylinder surface surrounding negative line charge $-\rho_l$ is

$$\varphi_- = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a}{d}$$

Then, the capacitance per unit length can be calculated as

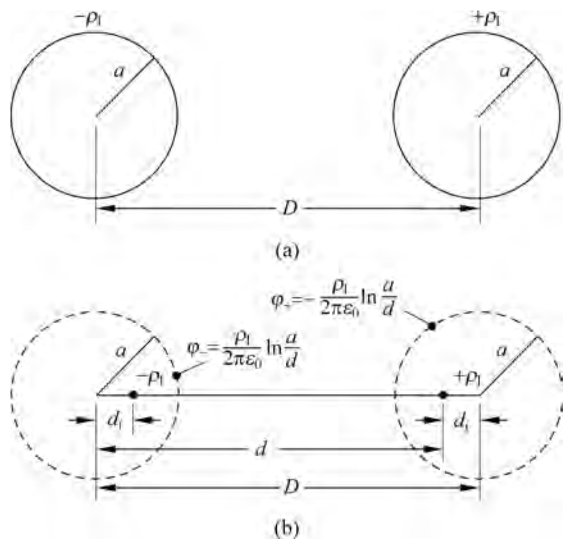

Figure 3-9 Two-wire transmission line and the equivalent line charges

$$C = \frac{\rho_l}{\varphi_+ - \varphi_-} = \frac{\pi\varepsilon_0}{\ln(d/a)} \tag{3.67}$$

Substituting (3.66) into (3.67) we have

$$C = \frac{\pi\varepsilon_0}{\ln[(D/2a) + \sqrt{(D/2a)^2 - 1}]} \tag{3.68}$$

Since

$$\ln[x + \sqrt{x^2 - 1}] = \cosh^{-1}x$$

for $x>1$, (3.68) can be written alternatively as

$$C = \frac{\pi\varepsilon_0}{\cosh^{-1}(D/2a)} \tag{3.69}$$

The potential distribution and electric field intensity around the two-wire line can be determined easily from the equivalent line charges.

The more general case of a two-wire transmission line of different radii can also be solved by using the method of image in a similar way. The key is to find the location of the equivalent line charges that make the wire surfaces equipotential.

3.5 Method of Separation of Variables(分离变量法)

The method of images is useful in solving certain types of electrostatic problems in which conducting boundaries can be replaced by equivalent charges. However, when the geometry of the boundaries is not simple, and/or the free charges are not known, the method of images cannot be used. In some problems, a system of conductors is maintained at specified potentials or specified normal derivatives of the potentials. If the boundaries of the conductors coincide with the coordinate surfaces of an orthogonal coordinate system, we may solve the problem by using the **method of separation of variables** (**分离变量法**).

In this section, the method of separation of variables is introduced as a method of solving Laplace's equations with given boundary conditions of the potential φ. Generally, problems formulated as partial differential equations with prescribed boundary conditions are called **boundary-value problems**(**边界值问题**). Boundary-value problems for electrostatic potential functions can be classified into three types: ① **Dirichlet problems**(**狄里赫利问题**,第一类边值问题), in which the value of the potential is specified everywhere on the boundaries; ② **Neumann problems**(**纽曼问题**,第二类边值问题), in which the normal derivative of the potential is specified everywhere on the boundaries; ③ **Mixed boundary-value problems**(**混合边值问题**), in which the potential is specified over some boundaries and the normal derivative of the potential is specified over the remaining ones. Different specified boundary conditions will require the choice of different potential functions, as will be demonstrated in this section. The solutions of Laplace's equation are often called harmonic functions(调和函数).①

Laplace's equation for scalar electric potential φ in Cartesian coordinates is

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0 \tag{3.70}$$

To apply the method of separation of variables, we assume that the solution $\varphi(x,y,z)$ can be expressed as a product in the following form:

$$\varphi(x,y,z)=X(x)Y(y)Z(z) \tag{3.71}$$

where $X(x)$, $Y(y)$, and $Z(z)$ are functions of only x, y and z, respectively. Substituting (3.71) in (3.70), we have

$$Z(z)Y(y)\frac{\mathrm{d}^2X(x)}{\mathrm{d}x^2}+X(x)Z(z)\frac{\mathrm{d}^2Y(y)}{\mathrm{d}y^2}+X(x)Y(y)\frac{\mathrm{d}^2Z(z)}{\mathrm{d}z^2}=0$$

Divide both sides of the above equation by the product $X(x)Y(y)Z(z)$, we have

$$\frac{1}{X(x)}\frac{\mathrm{d}^2X(x)}{\mathrm{d}x^2}+\frac{1}{Y(y)}\frac{\mathrm{d}^2Y(y)}{\mathrm{d}y^2}+\frac{1}{Z(z)}\frac{\mathrm{d}^2Z(z)}{\mathrm{d}z^2}=0 \tag{3.72}$$

Notice that each of the three terms on the left side of (3.72) is a function of only one coordinate variable. In order for (3.72) to be satisfied for all values of x,y,z, each of the three terms must be a constant. For instance, if we differentiate (3.72) with respect to x, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{X(x)}\frac{\mathrm{d}^2X(x)}{\mathrm{d}x^2}\right]=0 \tag{3.73}$$

This requires that

$$\frac{1}{X(x)}\frac{\mathrm{d}^2X(x)}{\mathrm{d}x^2}=-k_x^2 \tag{3.74}$$

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① 分离变量法是求解边值问题的一种经典方法,其基本思想是将偏微分方程中含有 n 个自变量的待求函数表示成 n 个只含一个变量的函数的乘积,把偏微分方程分解成 n 个常微分方程,求出各常微分方程的通解后,将它们线性叠加得到级数形式解,并利用给定的边界条件确定待定常数。分离变量法的理论依据是唯一性定理。

where k_x^2 is a constant of integration that will be determined later from the boundary conditions of the problem. The negative sign on the right side of(3.74) as well as the square sign on k_x are employed only for mathematical convenience, which will be seen later. The separation constant k_x can be a real or an imaginary number. If k_x is imaginary, k_x^2 is a negative real number, making $-k_x^2$ a positive real number. Now we rewrite(3.74) as

$$\frac{d^2X(x)}{dx^2}+k_x^2X(x)=0 \tag{3.75}$$

Similarly, we can obtain equations for the functions $Y(y)$ and $Z(z)$ as

$$\frac{d^2Y(y)}{dy^2}+k_y^2Y(y)=0 \tag{3.76}$$

$$\frac{d^2Z(z)}{dz^2}+k_z^2Z(z)=0 \tag{3.77}$$

where the separation constants k_y and k_z are generally different from k_x, but due to (3.72), satisfy:

$$k_x^2+k_y^2+k_z^2=0 \tag{3.78}$$

Now the problem is reduced to finding the appropriate solutions $X(x)$, $Y(y)$ and $Z(z)$ from the second-order ordinary differential equations (3.75), (3.76) and (3.77) respectively. The possible solutions of(3.75) are well known and listed in Table 3-1.

Table 3-1 Solutions of equation $X''(x)+k_x^2X(x)=0$ ①

k_x	$X(x)$	Exponential form of $X(x)$
0	A_0x+B_0	
k	$A_1\sin kx+B_1\cos kx$	$C_1e^{jkx}+D_1e^{-jkx}$
jk	$A_2\sinh kx+B_2\cosh kx$	$C_2e^{kx}+D_2e^{-kx}$

The first possible solution, $X(x)=A_0x+B_0$, as listed in Table 3-1 is the result of $k_x=0$, in which case the potential function is a straight line with a slope A_0 and an intercept B_0 at $x=0$. Example 3-1 is an example of this solution.

When $k_x=k$ is a nonzero real number, the solution is a linear combination of $\sin kx$ and $\cos kx$, both of which have a period of $2\pi/k$. Generally, if the potential to be solved is periodic (usually with multiple zeros) along x-direction, a linear combination of $\sin kx$ and $\cos kx$ should be chosen as the solution. In some special cases, if the boundary condition requires the potential to be zero at $x=0$, $\sin kx$ alone must be chosen; if the potential is expected to be symmetrical with respect to $x=0$, then $\cos kx$ alone must be chosen. Sometimes it may be desirable to use

① 表 3-1 中列出的指数函数、三角函数以及双曲函数之间有如下转换关系：

$$e^{\pm jkx}=\cos kx\pm\sin kx,\cos kx=\frac{1}{2}(e^{jkx}+e^{-jkx}),\sin kx=\frac{1}{2j}(e^{jkx}-e^{-jkx})$$

$$e^{\pm kx}=\cosh kx\pm\sinh kx,\cosh kx=\frac{e^{kx}+e^{-kx}}{2},\sinh kx=\frac{1}{2}(e^{kx}-e^{-kx})$$

具体采用哪类函数作为方程的解取决于具体问题的边界条件。

$A_1\sin k(x-x_0)$ as the solution if a zero is found to be at $x=x_0$, whereas $B_1\cos k(x-x_0)$ should be used if the potential is symmetrical with respect to $x=x_0$.

If $k_x = jk$ is a purely imaginary number, the solution can take the form of a linear combination of hyperbolic functions $A_2\sinh kx+B_2\cosh kx$, or equivalently, exponential functions $C_2e^{kx}+D_2e^{-kx}$.

Hyperbolic and exponential functions are also plotted in Figure 3-10 for easy reference. These functions are non-periodic. The function $\sinh kx$ is an odd function of x and its value approaches $\pm\infty$ as x goes to $\pm\infty$. The function $\cosh kx$ is an even function of x. It equals unity at $x=0$, and approaches $+\infty$ as x goes to $+\infty$ or $-\infty$. The function e^{kx} approaches zero as x goes to $-\infty$ and approaches $+\infty$ as x goes to $+\infty$. The function e^{-kx} approaches $+\infty$ as x goes to $-\infty$ and approaches zero as x goes to $+\infty$.

The solutions of the equations (3.76) and (3.77) for $Y(y)$ and $Z(z)$ are similar. The choice of the proper form of the solution and the associated constants are determined by specified boundary conditions in specific problems, as shown in the following examples.

Example 3-8 As illustrated in Figure 3-11, two semi-infinite grounded conducting plates are parallel to the x-z plane and separated by a distance b. A third conducting plate in the y-z plane is insulated from the two grounded plates and maintained at a constant potential U_0. Determine the potential distribution in the region ($x>0, 0<y<b$) enclosed by the three plates.

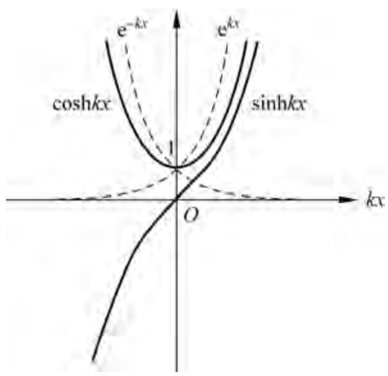

Figure 3-10 Illustration of different solutions of equation $X(x)+k_x^2 X(x)=0$

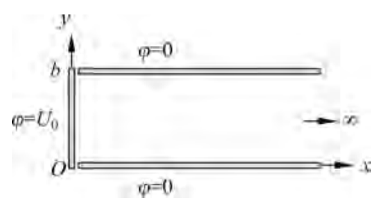

Figure 3-11 Illustration of electrostatic problem in Example 3-8

Solution: Referring to the coordinates in Figure 3-11, φ is independent of z, so we have

$$\varphi(x,y,z)=\varphi(x,y)=X(x)Y(y) \tag{3.79}$$

The boundary conditions for the potential are:

In the x-direction:

$$\varphi(0,y)=U_0 \tag{3.80a}$$

$$\varphi(\infty,y)=0 \tag{3.80b}$$

In the y-direction:

$$\varphi(x,0)=0 \tag{3.80c}$$

$$\varphi(x,b)=0 \tag{3.80d}$$

(3.79) implies that $k_z=0$ and from(3.78), we have

$$k_x^2+k_y^2=0 \tag{3.81}$$

We first notice that, according to the boundary condition(3.80b), $X(x)$ should approach zero as x approaches $+\infty$. Of all the possible solutions in Table 3-1, only the exponential function e^{-kx} meets this requirement, hence we can determine that $k_x=jk$ is imaginary and

$$X(x)=D_2e^{-kx} \tag{3.82}$$

where k is a real number. This choice of k_x implies that $k_y=k$ is real. So the function $Y(y)$ should be a combination of sine and cosine functions. Condition (3.80c) indicates that the proper choice for $Y(y)$ is

$$Y(y)=A_1\sin ky \tag{3.83}$$

Substitute (3.82) and (3.83) into (3.79), we obtain an appropriate solution of the following form:

$$\varphi(x,y)=(D_2A_1)e^{-kx}\sin ky=Ce^{-kx}\sin ky \tag{3.84}$$

where the product D_2A_1 has been combined into a single arbitrary constant C. Since (3.84) should satisfy(3.80d), we have,

$$\varphi(x,b)=Ce^{-kx}\sin kb=0 \tag{3.85}$$

which can be satisfied, for all values of x, only if

$$\sin kb=0$$

Therefore,

$$k=\frac{n\pi}{b},\quad n=1,2,3,\cdots \tag{3.86}$$

which means that k can only take discrete values. Substitute(3.86) into(3.84), we obtain

$$\varphi_n(x,y)=C_ne^{-n\pi x/b}\sin\frac{n\pi}{b}y \tag{3.87}$$

where the subscript n indicates the nth possible value of the constant k, and hence indicates the nth possible solution of φ. (Question: why n cannot be 0 or negative integral values?) Apparently, for any n value, the function $\varphi_n(x,y)$ in(3.87) satisfies the Laplace's equation and the boundary conditions (3.80b-d). But any $\varphi_n(x,y)$ alone cannot satisfy the remaining boundary condition(3.80a) at $x=0$ for all values of y from 0 to b. Nevertheless, since Laplace's equation is a linear partial differential equation, a linear combination of $\varphi_n(x,y)$ with all possible n values is also a solution, which could satisfy the boundary condition(3.80a). So, the desired solution can be written as

$$\varphi(x,y)=\sum_{n=1}^{\infty}\varphi_n(x,y)=\sum_{n=1}^{\infty}C_ne^{-n\pi x/b}\sin\frac{n\pi}{b}y \tag{3.88}$$

It is easy to verify that $\varphi(x,y)$ in(3.88) satisfies boundary conditions(3.80b-d). So now, we only need to let $\varphi(x,y)$ in(3.88) satisfy boundary condition(3.80a). This requires

$$\varphi(0,y)=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi}{b}y=U_0,\quad \text{for}\quad 0<y<b \tag{3.89}$$

(3.89) is essentially a Fourier-series expansion of the periodic rectangular wave with a fundamental period of $2b$ shown in Figure 3-12, which has a constant value U_0 in the interval $0<y<b$. Notice that the $\sin(n\pi/b)y$ term in (3.89) is an odd function, and therefore, the rectangular wave has a constant value $-U_0$ in the interval $-b<y<0$.

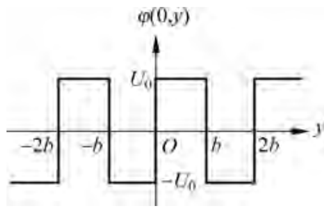

Figure 3-12 A periodic rectangular wave function

In order to evaluate the coefficients C_n, we multiply both sides of (3.89) by $\sin(m\pi/b)y$ and integrate the products from $y=0$ to $y=b$:

$$\sum_{n=1}^{\infty}\int_0^b C_n \sin\frac{n\pi}{b}y \sin\frac{m\pi}{b}y\,dy = \int_0^b U_0 \sin\frac{m\pi}{b}y\,dy \tag{3.90}$$

The integral on the right side of (3.90) is easily evaluated:

$$\int_0^b U_0 \sin\frac{m\pi}{b}y\,dy = \begin{cases}\dfrac{2bU_0}{m\pi} & \text{if } m \text{ is odd}\\ 0 & \text{if } m \text{ is even}\end{cases} \tag{3.91}$$

Each integral on the left side of (3.90) is

$$\begin{aligned}\int_0^b C_n \sin\frac{n\pi}{b}y \sin\frac{m\pi}{b}y\,dy &= \frac{C_n}{2}\int_0^b\left[\cos\frac{(n-m)\pi}{b}y - \cos\frac{(n+m)\pi}{b}y\right]dy\\ &= \begin{cases}\dfrac{C_n}{2}b & \text{if } m=n\\ 0 & \text{if } m\neq n\end{cases}\end{aligned} \tag{3.92}$$

Substituting (3.91) and (3.92) into (3.90), we obtain

$$\frac{C_m}{2}b = \begin{cases}\dfrac{2bU_0}{m\pi} & \text{if } m \text{ is odd}\\ 0 & \text{if } m \text{ is even}\end{cases}$$

which gives us the solution (with the index m replaced by n)

$$C_n = \begin{cases}\dfrac{4U_0}{n\pi} & \text{if } n \text{ is odd}\\ 0 & \text{if } n \text{ is even}\end{cases} \tag{3.93}$$

Substitute (3.93) into (3.88), we have the final solution of the potential distribution

$$\varphi(x,y) = \sum_{n=1,3,5,\cdots}^{\infty}\frac{4U_0}{n\pi}e^{-n\pi x/b}\sin\frac{n\pi}{b}y \quad \text{for} \quad x>0, 0<y<b \tag{3.94}$$

The solution (3.94) is a rather complicated expression involving a summation of infinite series. However, since the terms in the series decreases as $1/n$ as n increases, only the first few terms are needed to obtain a good approximation.

Example 3-9 Consider a region enclosed by four conducting plates as illustrated in Figure 3-13. The top, right and bottom plates are grounded. The left plate is insulated from the others and maintaned at a constant potential U_0. All plates are infinite in extent in the z-direction. Determine the potential distribution within this region.

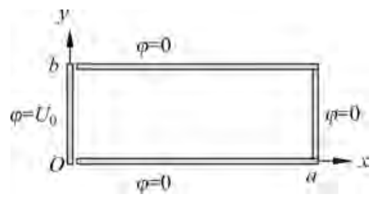

Figure 3-13 Illustration of electrostatic problem in Example 3-9

Solution: Like Example 3-8, the potential φ is independent of z, so we have

$$\varphi(x,y,z) = \varphi(x,y) = X(x)Y(y) \tag{3.95}$$

The boundary conditions are:

In the x-direction:

$$\varphi(0,y) = U_0 \tag{3.96a}$$

$$\varphi(a,y) = 0 \tag{3.96b}$$

In the y-direction:

$$\varphi(x,0) = 0 \tag{3.96c}$$

$$\varphi(x,b) = 0 \tag{3.96d}$$

(3.95) implies that $k_z = 0$ and from (3.78),

$$k_x^2 + k_y^2 = 0 \tag{3.97}$$

which is the same as (3.81) in Example 3-8. The boundary conditions in the y-direction, (3.96c) and (3.96d), are also the same as those specified in Example 3-8. To make $\varphi(x,0) = 0$ and $\varphi(x,b) = 0$ for all values of x between 0 and a, $Y(0)$ and $Y(b)$ must be zero. Of the functions listed in Table 3-1, only sine and cosine functions are periodic with multiple zeros, so $Y(y)$ must be a linear combination of sine and cosine functions. With $Y(0) = Y(b) = 0$, we have

$$Y(y) = A_1 \sin ky \tag{3.98}$$

which is the same as in (3.83), and k can take discrete values as

$$k = \frac{n\pi}{b}, \quad n = 1,2,3,\cdots \tag{3.99}$$

This means $k_y = k$ is real, and according to (3.97), $k_x = jk$. As a result, in the x-direction, $X(x)$ is a linear combination of sinh and cosh functions, i.e.,

$$X(x) = A_2 \sinh kx + B_2 \cosh kx \tag{3.100}$$

To determine A_2 and B_2, we apply the boundary condition (3.96b), which demands that $X(a) = 0$; that is,

$$0 = A_2 \sinh ka + B_2 \cosh ka$$

or

$$B_2 = -A_2 \frac{\sinh ka}{\cosh ka}$$

Therefore, we have

$$X(x) = A_2\left(\sinh kx - \frac{\sinh ka}{\cosh ka}\cosh kx\right) = \frac{A_2}{\cosh ka}(\cosh ka \sinh kx - \sinh ka \cosh kx)$$
$$= A_3 \sinh k(x-a) \qquad (3.101)$$

where $A_3 = A_2/\cosh ka$. Note that (3.101) is a shift in the argument of the sinh function. Now, we obtain the appropriate product solution

$$\varphi_n(x,y) = B_0 A_1 A_3 \sinh k(x-a) \sin ky = C'_n \sinh \frac{n\pi}{b}(x-a) \sin \frac{n\pi}{b} y \qquad (3.102)$$

where $C'_n = B_0 A_1 A_3$. We have now used all of the boundary conditions except (3.96a), which may be satisfied by a Fourier-series expansion of $\varphi(0,y) = U_0$ over the interval from $y=0$ to $y=b$. We have

$$\sum_{n=1}^{\infty} \varphi_n(0,y) = -\sum_{n=1}^{\infty} C'_n \sinh \frac{n\pi}{b} a \sin \frac{n\pi}{b} y = U_0, \quad (0<y<b) \qquad (3.103)$$

We note that (3.103) is of the same form as (3.89), except that C'_n is replaced by $-C'_n \sinh(n\pi a/b)$. The values for the coefficient C'_n can then be written down from (3.93):

$$C'_n = \begin{cases} -\dfrac{4U_0}{n\pi \sinh(n\pi a/b)} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \qquad (3.104)$$

The potential solution is then the summation of $\varphi_n(x,y)$ in (3.102) with the coefficient C'_n given by (3.104), i.e.,

$$\varphi(x,y) = \frac{4U_0}{\pi} \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n \sinh(n\pi a/b)} \sinh \frac{n\pi}{b}(a-x) \sin \frac{n\pi}{b} y$$
$$(0<x<a \quad \text{and} \quad 0<y<b) \qquad (3.105)$$

The electric field distribution within the enclosure is obtained by the relation

$$\boldsymbol{E}(x,y) = -\nabla\varphi(x,y) = -\left(\boldsymbol{e}_x \frac{\partial}{\partial x} + \boldsymbol{e}_y \frac{\partial}{\partial y}\right)\varphi(x,y)$$

Summary

Concepts

Laplacian operator (拉普拉斯算子)

Poisson's equation (泊松方程)

Laplace's equation (拉普拉斯方程)

Laws & Theorems

Uniqueness theorem (唯一性定理)

Methods

Method of images(镜像法)　　Method of separation of variables(分离变量法)

Review Questions

3.1　写出简单媒质中静电场电位满足的泊松方程。对于一般的媒质,该泊松方程是什么形式?

3.2　什么是静电场的唯一性定理?

3.3　若已知给定区域内 $\nabla^2\varphi=0$,是否可以推断该区域内的电位为常数? 若否,还需要什么条件才能得出该结论?

3.4　无限长均匀线电荷对平行于该线电荷的导体圆柱面的镜像是什么?

3.5　点电荷对接地导体球面的镜像是什么?

3.6　简述应用分离变量法求解静电场问题的原理和基本步骤。分离变量法适合于求解哪些类型的静电场问题?

3.7　静电场问题的分离变量法中3个分离系数 k_x、k_y 和 k_z 能全为实数么? 能全为虚数么? 为什么?

Problems

3.1　A large parallel-plate capacitor with height d is filled with two layers of dielectric slabs. The dielectric constant of the layer between $z=0$ and $z=0.8d$ is ε_{r1} and the dielectric constant of the layer between $z=0.8d$ and $z=d$ is ε_{r2}. The bottom plate at $z=0$ is grounded and the top plate at $z=d$ has a constant potential U_0. Assuming negligible fringing effect, determine

(1) the potential φ and electric field $\boldsymbol{E}$ inside the capacitor,

(2) the surface charge densities on the top and bottom plates.

3.2　Prove that the potential φ due to a charge distribution given in (2.58) satisfies Poisson's equation.

3.3　Prove that, if a potential function φ satisfies Laplace's equation in a given region and φ is constant on the boundary of the region, then φ is constant throughout the region.

3.4　Prove that a potential function satisfying Laplace's equation in a given region possesses no maximum or minimum within the region.

3.5　A point charge Q exists at a distance d above a large grounded conducting plate. Determine

(1) the surface charge density ρ_s on the conducting plate,

(2) the total charge induced on the conducting plate.

3.6　A straight-line charge of ρ_l is parallel to and at a height h from the surface of an infinitely large grounded conducting plate. Referring to Figure 3-14, prove that the surface charge induced on the plane is

$$\rho_s = \frac{-\rho_1 h}{\pi(x^2 + h^2)}$$

3.7 Two semi-infinitely large conducting plates are located in the y-z plane and x-z plane respectively, as illustrated in Figure 3-15. A point charge of 200 mC is placed at point $A(1,3,0)$. Determine the electric potential and the electric field intensity at point $B(3,2,0)$.

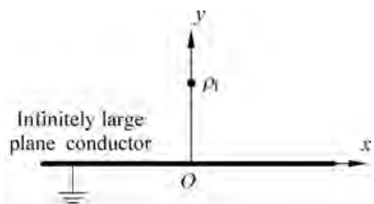

Figure 3-14 Illustration of Problem 3.6

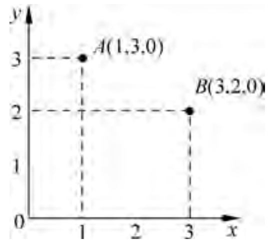

Figure 3-15 Illustration of Problem 3.7

3.8 Two semi-infinitely large grounded metal plates are located at $\phi = 0$ and $\phi = \pi/3$ respectively. A point charge q is situated at $\left(1, \frac{\pi}{6}, 0\right)$ in the cylindrical coordinate system. Find the potential at point $\left(3, \frac{\pi}{6}, 0\right)$.

3.9 A straight conducting wire of radius a is parallel to and at height h from the surface of the earth. Assuming that the earth is perfectly conducting, determine the capacitance between the wire and the earth.

3.10 A point charge Q resides inside a hollow spherical cavity with a grounded conducting shell. The radius of the cavity is a, and the point charge is at a distance d from the cavity center (where $a>d$). Use the method of images to determine ①the potential distribution inside the cavity, ②the charge density ρ_s induced on the inner surface of the shell.

3.11 A point charge Q is located at (x_0, y_0) outside a conducting hemisphere of radius a on top of an infinitely large conducting plate, as shown in Figure 3-16. Find the locations and values of the image charges that are needed for solving the fields outside the conductor.

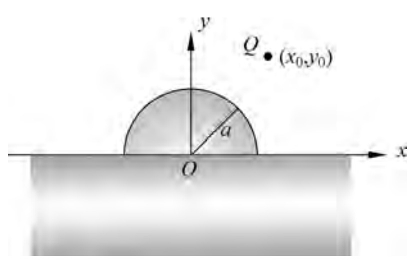

Figure 3-16 Illustration of Problem 3.11

3.12 Repeat solving the problem in Example 3-9 with the boundary conditions on the top, bottom, and right plates in Figure 3-13 changed to $\partial\varphi/\partial n = 0$.

3.13 For Example 3-9, if the top, bottom, and left plates in Figure 3-13 are grounded

($\varphi=0$) and the right plate is maintained at a constant potential U_0, prove that the potential distribution within the enclosed region is

$$\varphi(x,y)=\sum_{n=1}^{\infty}\frac{2U_0[1-(-1)^n]}{n\pi\sinh\left(\frac{n\pi a}{b}\right)}\sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b}$$

3.14 Consider the region enclosed by four conducting plates as shown in Figure 3-17 (the four plates are assumed to be infinitely long along the z-direction). The left and right plates are grounded, and the top and bottom plates have constant potentials U_1 and U_2 respectively. Find the potential distribution inside the enclosure.

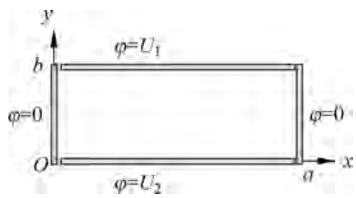

Figure 3-17 Illustration of Problem 3.14

3.15 Consider a metallic rectangular box with sides a and b and height c. The side walls and the bottom surface are grounded. The top surface is isolated and kept at a constant potential U_0. Determine the potential distribution inside the box.