1 Vectors and Matrices

Linear algebra is about vectors $\boldsymbol{v}$ and matrices A. Those are the basic objects that we can add and subtract and multiply (when their shapes match correctly). The first vector $\boldsymbol{v}$ has two components $\boldsymbol{v_1} = \mathbf{2}$ and $\boldsymbol{v_2} = \mathbf{4}$. The vector $\boldsymbol{w}$ is also 2-dimensional.

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \mathbf{2} \\ \mathbf{4} \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{3} \end{bmatrix} \qquad \boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} \mathbf{3} \\ \mathbf{7} \end{bmatrix}$$

The **linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ are the vectors** $c\boldsymbol{v} + d\boldsymbol{w}$ for all numbers c and d:

$$\textbf{The linear combinations} \quad \boldsymbol{c}\begin{bmatrix} \mathbf{2} \\ 4 \end{bmatrix} + \boldsymbol{d}\begin{bmatrix} \mathbf{1} \\ \mathbf{3} \end{bmatrix} = \begin{bmatrix} \mathbf{2}\boldsymbol{c} + \mathbf{1}\boldsymbol{d} \\ 4\boldsymbol{c} + \mathbf{3}\boldsymbol{d} \end{bmatrix} \quad \textbf{fill the } \boldsymbol{xy} \textbf{ plane}$$

The **length** of that vector $\boldsymbol{w}$ is $||\boldsymbol{w}|| = \sqrt{10}$, the square root of $w_1^2 + w_2^2 = 1 + 9$. The **dot product** of $\boldsymbol{v}$ and $\boldsymbol{w}$ is $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 = (\mathbf{2})(\mathbf{1}) + (\mathbf{4})(\mathbf{3}) = \mathbf{14}$. In Section 1.2, $\boldsymbol{v} \cdot \boldsymbol{w}$ will reveal the angle between those vectors.

The big step in Section 1.3 is to introduce a **matrix**. This matrix $\boldsymbol{A}$ contains our two column vectors. The vectors have two components, so the matrix is 2 by 2:

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{v} & \boldsymbol{w} \end{bmatrix} = \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ 4 & \mathbf{3} \end{bmatrix}.$$

When a matrix multiplies a vector, we get a combination $c\boldsymbol{v} + d\boldsymbol{w}$ of its columns:

$$\boldsymbol{A} \textbf{ times} \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ 4 & \mathbf{3} \end{bmatrix} \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} \mathbf{2}\boldsymbol{c} + \mathbf{1}\boldsymbol{d} \\ 4\boldsymbol{c} + \mathbf{3}\boldsymbol{d} \end{bmatrix} = c\boldsymbol{v} + d\boldsymbol{w}.$$

And when we look at **all combinations $\boldsymbol{Ax}$** (with every c and d), those vectors produce the **column space of the matrix A**. Here that column space is a plane.

With three vectors, the new matrix $\boldsymbol{B}$ has columns $\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{z}$. In this example $\boldsymbol{z}$ is a combination of $\boldsymbol{v}$ and $\boldsymbol{w}$. So the column space of $\boldsymbol{B}$ is still the xy plane. The vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are **independent** but $\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{z}$ are **dependent**. A combination produces zero:

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{v} & \boldsymbol{w} & \boldsymbol{z} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 7 \end{bmatrix} \text{ has } \boldsymbol{B} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{-1} \end{bmatrix} = \boldsymbol{v} + \boldsymbol{w} - \boldsymbol{z} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

The final goal is to understand **matrix multiplication $AB = \boldsymbol{A}$ times each column of $\boldsymbol{B}$**.

1.1 Vectors and Linear Combinations

1 $\boldsymbol{2v - 3w}$ is a **linear combination** $\boldsymbol{cv + dw}$ of the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$.

2 For $\boldsymbol{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ that combination is $2\begin{bmatrix} 4 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

3 All combinations $c\begin{bmatrix} 4 \\ 1 \end{bmatrix} + d\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ fill the xy plane. They produce every $\begin{bmatrix} x \\ y \end{bmatrix}$.

4 The vectors $c\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$ fill a **plane** in xyz space. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ **is not on that plane**.

Calculus begins with numbers x and functions $\boldsymbol{f}(x)$. Linear algebra begins with vectors $\boldsymbol{v}, \boldsymbol{w}$ and their linear combinations $c\boldsymbol{v} + d\boldsymbol{w}$. Immediately this takes you into two or more (possibly many more) dimensions. But linear combinations of vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are built from just two basic operations:

Multiply a vector $\boldsymbol{v}$ by a number $\quad \boldsymbol{3v} = \boldsymbol{3}\begin{bmatrix} \boldsymbol{2} \\ \boldsymbol{1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{6} \\ \boldsymbol{3} \end{bmatrix}$

Add vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ of the same dimension: $\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} \boldsymbol{2} \\ \boldsymbol{1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{4} \\ \boldsymbol{3} \end{bmatrix} = \begin{bmatrix} \boldsymbol{6} \\ \boldsymbol{4} \end{bmatrix}$

Those operations come together in a **linear combination** $c\boldsymbol{v} + d\boldsymbol{w}$ of $\boldsymbol{v}$ and $\boldsymbol{w}$:

Linear combination
$\boldsymbol{c = 5}$ and $\boldsymbol{d = -2}$ $\quad \boldsymbol{5v - 2w = 5}\begin{bmatrix} \boldsymbol{2} \\ \boldsymbol{1} \end{bmatrix} - \boldsymbol{2}\begin{bmatrix} \boldsymbol{4} \\ \boldsymbol{3} \end{bmatrix} = \begin{bmatrix} \boldsymbol{10} \\ \boldsymbol{5} \end{bmatrix} - \begin{bmatrix} \boldsymbol{8} \\ \boldsymbol{6} \end{bmatrix} = \begin{bmatrix} \boldsymbol{2} \\ \boldsymbol{-1} \end{bmatrix}$

That idea of a linear combination opens up two key questions:

1 **Describe all the combinations $\boldsymbol{cv + dw}$.** Do they fill a plane or a line?

2 **Find the numbers $\boldsymbol{c}$ and $\boldsymbol{d}$** that produce a specific combination $c\boldsymbol{v} + d\boldsymbol{w} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

We can answer those questions here in Section 1.1. But linear algebra is not limited to 2 vectors in 2-dimensional space. As long as we stay linear, the problems can get bigger (more dimensions) and harder (more vectors). The vectors will have m components instead of 2 components. We will have n vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ instead of 2 vectors. **Those $\boldsymbol{n}$ vectors in $\boldsymbol{m}$-dimensional space will go into the columns of an $\boldsymbol{m}$ by $\boldsymbol{n}$ matrix $\boldsymbol{A}$:**

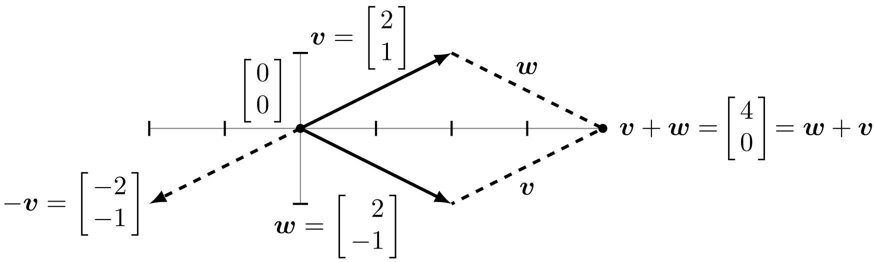
$\boldsymbol{m}$ rows
$\boldsymbol{n}$ columns $\quad \boldsymbol{A} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix}$
$\boldsymbol{m}$ by $\boldsymbol{n}$ matrix

Let me repeat the two key questions using A, and then retreat back to $m = 2$ and $n = 2$:

1	**Describe all the combinations** $A\boldsymbol{x} = x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \cdots + x_n\boldsymbol{v}_n$ **of the columns**
2	**Find the numbers $\boldsymbol{x_1}$ to $\boldsymbol{x_n}$** that produce a desired output vector $\boldsymbol{Ax = b}$

Linear Combinations $cv + dw$

Start from the beginning. A vector $\boldsymbol{v}$ in 2-dimensional space has *two components*. To draw $\boldsymbol{v}$ and $-\boldsymbol{v}$, use an arrow that begins at the zero vector:



The vectors $c\boldsymbol{v}$ (for all numbers c) fill an infinitely long line in the xy plane. If $\boldsymbol{w}$ is not on that line, then the vectors $d\boldsymbol{w}$ fill a second line. We aim to see that **the linear combinations $c\boldsymbol{v} + d\boldsymbol{w}$ fill the plane**. Combining points on the $\boldsymbol{v}$ line and the $\boldsymbol{w}$ line gives all points.

Here are four different linear combinations—*we can choose any numbers c and d*:

$$
\begin{array}{rcl}
1\boldsymbol{v} + 1\boldsymbol{w} & = & \text{sum of vectors} \\
1\boldsymbol{v} - 1\boldsymbol{w} & = & \text{difference of vectors} \\
0\boldsymbol{v} + 0\boldsymbol{w} & = & \textbf{\textit{zero vector}} \\
c\boldsymbol{v} + 0\boldsymbol{w} & = & \text{vector } c\boldsymbol{v} \text{ in the direction of } \boldsymbol{v}
\end{array}
$$

Solving Two Equations

Very often linear algebra offers the choice of a picture or a computation. The picture can start with $\boldsymbol{v} + \boldsymbol{w}$ and $\boldsymbol{w} + \boldsymbol{v}$ (those are equal). Another important vector is $\boldsymbol{w} - \boldsymbol{v}$, going backwards on $\boldsymbol{v}$. The Preface has a picture with many more combinations—starting to fill a plane. But if we aim for a particular vector like $c\boldsymbol{v} + d\boldsymbol{w} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$, it will be better to compute the exact numbers c and d. Here are two ways to write down this problem.

$$
\boxed{\text{Solve} \quad c\begin{bmatrix} 2 \\ 1 \end{bmatrix} + d\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}. \qquad \text{This means} \quad \begin{array}{r} \mathbf{2c + 2d = 8} \\ \mathbf{c - d = 2} \end{array}.}
$$

The rules for solution are simple but strict. We can multiply equations by numbers (*not zero!*) and we can subtract one equation from another equation. Experience teaches that the key is to produce zeros on the left side of the equations. One zero will be enough!

$\frac{1}{2}$(equation 1) is $\boldsymbol{c} + \boldsymbol{d} = \mathbf{4}$

Subtract this from equation 2

Then c is eliminated

$$
\begin{aligned}
\mathbf{2}\boldsymbol{c} + \mathbf{2}\boldsymbol{d} &= \mathbf{8} \\
\mathbf{0}\boldsymbol{c} - \mathbf{2}\boldsymbol{d} &= \mathbf{-2}
\end{aligned}
$$

The second equation gives $\boldsymbol{d} = \mathbf{1}$. Going upwards, the first equation becomes $\mathbf{2}\boldsymbol{c} + \mathbf{2} = \mathbf{8}$. Its solution is $\boldsymbol{c} = \mathbf{3}$. This combination is correct:

$$
\mathbf{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \mathbf{1}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}. \quad \text{In matrix form} \begin{bmatrix} \mathbf{2} & \mathbf{2} \\ \mathbf{1} & \mathbf{-1} \end{bmatrix}\begin{bmatrix} \mathbf{3} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{8} \\ \mathbf{2} \end{bmatrix}
$$

Column Way, Row Way, Matrix Way

Column way **Linear combination**	$c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + d\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$
Row way **Two equations for c and d**	$\begin{aligned} v_1c + w_1d &= b_1 \\ v_2c + w_2d &= b_2 \end{aligned}$
Matrix way **2 by 2 matrix**	$\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

If the points $\boldsymbol{v}$ and $\boldsymbol{w}$ and the zero vector $\mathbf{0}$ are not on the same line, there is exactly one solution $\boldsymbol{c}, \boldsymbol{d}$. Then the linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ exactly fill the xy plane. The vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are **"linearly independent"**. The 2 by 2 matrix $A = \begin{bmatrix} \boldsymbol{v} & \boldsymbol{w} \end{bmatrix}$ is "*invertible*".

Can Elimination Fail ?

Elimination fails to produce a solution only when the equations don't have a solution in the first place. This can happen when the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ *lie on the same line through the center point* $(0, 0)$. Those vectors are not independent.

The reason is clear: All combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ will then *lie on that same line*. If the desired vector $\boldsymbol{b}$ is *off the line*, then the equations $c\boldsymbol{v} + d\boldsymbol{w} = \boldsymbol{b}$ have no solution:

Example $\quad \boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{aligned} \mathbf{1}\boldsymbol{c} + \mathbf{3}\boldsymbol{d} = \mathbf{1} \\ \mathbf{2}\boldsymbol{c} + \mathbf{6}\boldsymbol{d} = \mathbf{0} \end{aligned}$

Those two equations can't be solved. To eliminate the 2 in the second equation, we multiply equation 1 by 2. Then elimination subtracts $2c + 6d = \mathbf{2}$ from the second equation $2c + 6d = \mathbf{0}$. **The result is $\mathbf{0} = -\mathbf{2}$:** *impossible*. We not only eliminated $\boldsymbol{c}$, we also eliminated $\boldsymbol{d}$.

With $\boldsymbol{v}$ and $\boldsymbol{w}$ on the same line, combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ fill that line but not a plane. When $\boldsymbol{b}$ is not on that line, no combination of $\boldsymbol{v}$ and $\boldsymbol{w}$ equals $\boldsymbol{b}$. The original vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are **"linearly dependent"** because $\boldsymbol{w} = 3\boldsymbol{v}$.

Linear combinations of two *independent vectors* $\boldsymbol{v}$ and $\boldsymbol{w}$ in two-dimensional space can produce any vector $\boldsymbol{b}$ in that plane. Then these equations have a solution:

2 equations
2 unknowns $\qquad c\boldsymbol{v} + d\boldsymbol{w} = \boldsymbol{b} \qquad \begin{aligned} cv_1 + dw_1 = b_1 \\ cv_2 + dw_2 = b_2 \end{aligned}$

Summary The combinations $c\boldsymbol{v} + d\boldsymbol{w}$ fill the x-y plane unless $\boldsymbol{v}$ is in line with $\boldsymbol{w}$.

Important Here is a different example where elimination seems to be in trouble. But we can easily fix the problem and go on.

$$c\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} + d\begin{bmatrix} \mathbf{2} \\ \mathbf{3} \end{bmatrix} = \begin{bmatrix} \mathbf{2} \\ \mathbf{7} \end{bmatrix} \quad \text{or} \quad \begin{aligned} 0 + 2d = 2 \\ c + 3d = 7 \end{aligned}$$

That zero looks dangerous. But we only have to **exchange equations to find $\boldsymbol{d}$ and $\boldsymbol{c}$**:

$$\begin{aligned} c + 3d = 7 \\ 0 + 2d = 2 \end{aligned} \quad \text{leads to} \quad \boldsymbol{d} = 1 \text{ and } \boldsymbol{c} = 4$$

Vectors in Three Dimensions

Suppose $\boldsymbol{v}$ and $\boldsymbol{w}$ have three components instead of two. Now they are vectors in three-dimensional space (which we will soon call $\mathbf{R}^3$). We still think of points in the space and arrows out from the zero vector. And we still have linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$:

$$\boldsymbol{v}=\begin{bmatrix}2\\3\\1\end{bmatrix}\qquad \boldsymbol{w}=\begin{bmatrix}1\\1\\0\end{bmatrix}\qquad \boldsymbol{v}+\boldsymbol{w}=\begin{bmatrix}3\\4\\1\end{bmatrix}\qquad c\boldsymbol{v}+d\boldsymbol{w}=\begin{bmatrix}2c+d\\3c+d\\c+0\end{bmatrix}$$

But there is a difference ! The combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ **do not fill** the whole 3-dimensional space. If we only have 2 vectors, their combinations can at most fill a 2-dimensional plane. It is not a case of linear dependence, it is just a case of not enough vectors.

We need *three* independent vectors if we want their combinations to fill 3-dimensional space. Here is the simplest choice $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$ for those three independent vectors:

$$\boldsymbol{i}=\begin{bmatrix}1\\0\\0\end{bmatrix}\qquad \boldsymbol{j}=\begin{bmatrix}0\\1\\0\end{bmatrix}\qquad \boldsymbol{k}=\begin{bmatrix}0\\0\\1\end{bmatrix}\qquad c\boldsymbol{i}+d\boldsymbol{j}+e\boldsymbol{k}=\begin{bmatrix}c\\d\\e\end{bmatrix}\qquad (1)$$

Now $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$ go along the x,y,z axes in three-dimensional space $\mathbf{R}^3$. We can easily write any vector $\boldsymbol{v}$ in $\mathbf{R}^3$ as a combination of $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$:

Vector form
Matrix form

$$\boldsymbol{v}=\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=v_1\boldsymbol{i}+v_2\boldsymbol{j}+v_3\boldsymbol{k}\qquad \begin{bmatrix}\mathbf{1}&0&0\\0&\mathbf{1}&0\\0&0&\mathbf{1}\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$$

That is the 3 by 3 **identity matrix** $\boldsymbol{I}$. Multiplying by I leaves every vector unchanged. I is the matrix analog of the number 1, because $I\boldsymbol{v}=\boldsymbol{v}$ for every $\boldsymbol{v}$.

Notice that we could not take a linear combination of a 2-dimensional vector $\boldsymbol{v}$ and a 3-dimensional vector $\boldsymbol{w}$. They are not in the same space.

How Do We Know It is a Plane ?

Suppose $\boldsymbol{v}$ and $\boldsymbol{w}$ are nonzero vectors with three components each. Assume they are *independent*, so the vectors point in different directions: $\boldsymbol{w}$ is not a multiple $c\boldsymbol{v}$. *Then their linear combinations fill a plane inside* 3*-dimensional space*. The surface is flat. Here is one way to see that this is true:

> Look at any two combinations $c\boldsymbol{v}+d\boldsymbol{w}$ and $C\boldsymbol{v}+D\boldsymbol{w}$. Halfway between those points is $\boldsymbol{h}=\frac{1}{2}(c+C)\boldsymbol{v}+\frac{1}{2}(d+D)\boldsymbol{w}$. This is another combination of $\boldsymbol{v}$ and $\boldsymbol{w}$. So our surface has the basic property of a plane: Halfway between any two points on the surface is **another point $\boldsymbol{h}$ on the surface**. The surface must be flat !
>
> Maybe that reasoning is not complete, even with the exclamation point. We depended on intuition for the properties of a plane. Another proof is coming in Section 1.2.

Problem Set 1.1

1 Under what conditions on a, b, c, d is $\begin{bmatrix} c \\ d \end{bmatrix}$ a multiple m of $\begin{bmatrix} a \\ b \end{bmatrix}$? Start with the two equations $c = ma$ and $d = mb$. By eliminating m, find one equation connecting a, b, c, d. You can assume no zeros in these numbers.

2 Going around a triangle from $(0, 0)$ to $(5, 0)$ to $(0, 12)$ to $(0, 0)$, what are those three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$? What is $\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}$? What are their lengths $||\boldsymbol{u}||$ and $||\boldsymbol{v}||$ and $||\boldsymbol{w}||$? The length squared of a vector $\boldsymbol{u} = (u_1, u_2)$ is $||\boldsymbol{u}||^2 = u_1^2 + u_2^2$.

Problems 3–10 are about addition of vectors and linear combinations.

3 Describe geometrically (line, plane, or all of $\mathbf{R}^3$) all linear combinations of

$$\text{(a)} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \qquad \text{(b)} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \qquad \text{(c)} \quad \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

4 Draw $\boldsymbol{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ and $\boldsymbol{v} + \boldsymbol{w}$ and $\boldsymbol{v} - \boldsymbol{w}$ in a single xy plane.

5 If $\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} - \boldsymbol{w} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, compute and draw the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$.

6 From $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find the components of $3\boldsymbol{v} + \boldsymbol{w}$ and $c\boldsymbol{v} + d\boldsymbol{w}$.

7 Compute $\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}$ and $2\boldsymbol{u} + 2\boldsymbol{v} + \boldsymbol{w}$. How do you know $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ lie in a plane?

These lie in a plane because $\boldsymbol{w} = c\boldsymbol{u} + d\boldsymbol{v}$. Find c and d

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}.$$

8 Every combination of $\boldsymbol{v} = (1, -2, 1)$ and $\boldsymbol{w} = (0, 1, -1)$ has components that add to _____. Find c and d so that $c\boldsymbol{v} + d\boldsymbol{w} = (3, 3, -6)$. Why is $(3, 3, 6)$ impossible?

9 In the xy plane mark all nine of these linear combinations:

$$c \begin{bmatrix} 2 \\ 1 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{with} \quad c = 0, 1, 2 \quad \text{and} \quad d = 0, 1, 2.$$

10 (Not easy) How could you decide if the vectors $\boldsymbol{u} = (1, 1, 0)$ and $\boldsymbol{v} = (0, 1, 1)$ and $\boldsymbol{w} = (a, b, c)$ are linearly independent or dependent ?

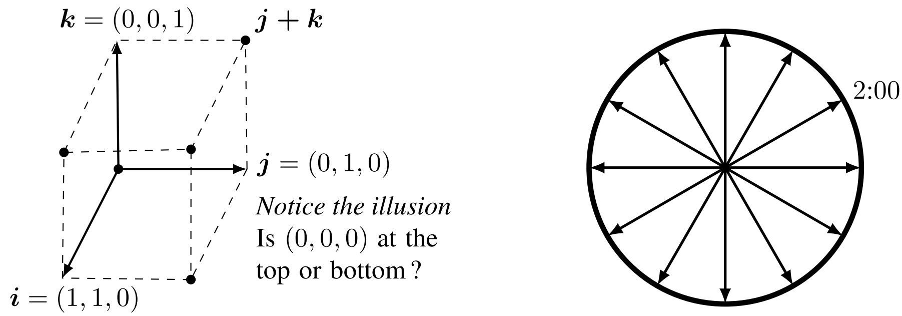


Figure 1.1: Unit cube from $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ and twelve clock vectors : all lengths $= 1$.

11 If three corners of a parallelogram are $(1, 1)$, $(4, 2)$, and $(1, 3)$, what are all three of the possible fourth corners? Draw those three parallelograms.

Problems 12–15 are about special vectors on cubes and clocks in Figure 1.1.

12 Four corners of this unit cube are $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. What are the other four corners? Find the coordinates of the center point of the cube. The center points of the six faces have coordinates _____. The cube has how many edges ?

13 *Review Question*. In xyz space, where is the plane of all linear combinations of $\boldsymbol{i} = (1, 0, 0)$ and $\boldsymbol{i} + \boldsymbol{j} = (1, 1, 0)$?

14 (a) What is the sum $\boldsymbol{V}$ of the twelve vectors that go from the center of a clock to the hours 1:00, 2:00, ..., 12:00?

(b) If the 2:00 vector is removed, why do the 11 remaining vectors add to 8:00?

(c) The components of that 2:00 vector are $\boldsymbol{v} = (\cos\theta, \sin\theta)$? What is θ ?

15 Suppose the twelve vectors start from 6:00 at the bottom instead of $(0, 0)$ at the center. The vector to 12:00 is doubled to $(0, 2)$. The new twelve vectors add to ____.

16 Draw vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ so that their combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ fill only a line. Find vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ in 3D so that their combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ fill only a plane.

17 What combination $c \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \end{bmatrix} + d \begin{bmatrix} \mathbf{3} \\ \mathbf{1} \end{bmatrix}$ produces $\begin{bmatrix} \mathbf{14} \\ \mathbf{8} \end{bmatrix}$? Express this question as two equations for the coefficients c and d in the linear combination.

Problems 18–19 go further with linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ (see Figure 1.2a).

18 Figure 1.2a shows $\frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$. Mark the points $\frac{3}{4}\boldsymbol{v} + \frac{1}{4}\boldsymbol{w}$ and $\frac{1}{4}\boldsymbol{v} + \frac{1}{4}\boldsymbol{w}$ and $\boldsymbol{v} + \boldsymbol{w}$. Draw the line of all combinations $c\boldsymbol{v} + d\boldsymbol{w}$ that have $c + d = 1$.

19 Restricted by $0 \le c \le 1$ and $0 \le d \le 1$, shade in all the combinations $c\boldsymbol{v} + d\boldsymbol{w}$. Restricted only by $c \ge 0$ and $d \ge 0$ draw the “cone” of all combinations $c\boldsymbol{v} + d\boldsymbol{w}$.

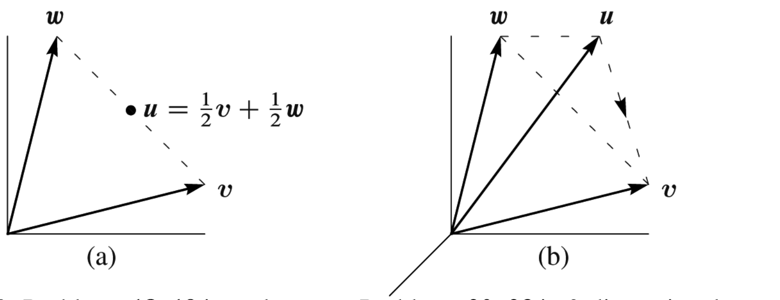


Figure 1.2: Problems **18–19** in a plane Problems **20–23** in 3-dimensional space

Problems 20–23 deal with $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in three-dimensional space (see Figure 1.2b).

20 Locate $\frac{1}{3}\boldsymbol{u} + \frac{1}{3}\boldsymbol{v} + \frac{1}{3}\boldsymbol{w}$ and $\frac{1}{2}\boldsymbol{u} + \frac{1}{2}\boldsymbol{w}$ in Figure 1.2 b. Challenge problem: Under what restrictions on c, d, e, will the combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ fill in the dashed triangle? To stay in the triangle, one requirement is $c \geq 0, d \geq 0, e \geq 0$.

21 The three dashed lines in the triangle are $\boldsymbol{v} - \boldsymbol{u}$ and $\boldsymbol{w} - \boldsymbol{v}$ and $\boldsymbol{u} - \boldsymbol{w}$. Their sum is _____. Draw the head-to-tail addition around a plane triangle of $(3, 1)$ plus $(-1, 1)$ plus $(-2, -2)$.

22 Shade in the pyramid of combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ with $c \geq 0$, $d \geq 0$, $e \geq 0$ and $c + d + e \leq 1$. Mark the vector $\frac{1}{2}(\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w})$ as inside or outside this pyramid.

23 If you look at *all* combinations of those $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{w}$, is there any vector that can't be produced from $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$? Different answer if $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are all in _____.

Challenge Problems

24 How many corners $(\pm1, \pm1, \pm1, \pm1)$ does a cube of side 2 have in 4 dimensions? What is its volume? How many 3D faces? How many edges? Find one edge.

25 Find *two different combinations* of the three vectors $\boldsymbol{u} = (1, 3)$ and $\boldsymbol{v} = (2, 7)$ and $\boldsymbol{w} = (1, 5)$ that produce $\boldsymbol{b} = (0, 1)$. Slightly delicate question: If I take any three vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ in the plane, will there always be two different combinations that produce $\boldsymbol{b} = (0, 1)$?

26 The linear combinations of $\boldsymbol{v} = (a, b)$ and $\boldsymbol{w} = (c, d)$ fill the plane unless _____. Find four vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, $\boldsymbol{z}$ with four nonzero components each so that their combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} + f\boldsymbol{z}$ produce all vectors in four-dimensional space.

27 Write down three equations for c, d, e so that $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} = \boldsymbol{b}$. Write this also as a matrix equation $A\boldsymbol{x} = \boldsymbol{b}$. Can you somehow find c, d, e for this $\boldsymbol{b}$?

$$\boldsymbol{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

1.2 Lengths and Angles from Dot Products

1 The "dot product" of $\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ is $\boldsymbol{v} \cdot \boldsymbol{w} = (1)(4) + (2)(6) = 4 + 12 = \mathbf{16}$.

2 The length squared of $\boldsymbol{v} = (1, 3, 2)$ is $\boldsymbol{v} \cdot \boldsymbol{v} = 1 + 9 + 4 = 14$. **The length is** $||\boldsymbol{v}|| = \sqrt{\mathbf{14}}$.

3 $\boldsymbol{v} = (1, 3, 2)$ is **perpendicular** to $\boldsymbol{w} = (4, -4, 4)$ because $\boldsymbol{v} \cdot \boldsymbol{w} = \mathbf{0}$.

4 The angle $\theta = 45^\circ$ between $\boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has $\cos\theta = \dfrac{\boldsymbol{v} \cdot \boldsymbol{w}}{||\boldsymbol{v}||\ ||\boldsymbol{w}||} = \dfrac{1}{(1)(\sqrt{2})}$.

5 All angles have $|\cos\theta| \leq 1$. All vectors have **Schwarz inequality** $|\boldsymbol{v} \cdot \boldsymbol{w}| \leq ||\boldsymbol{v}||\ ||\boldsymbol{w}||$ | **Triangle inequality** $||\boldsymbol{v}+\boldsymbol{w}|| \leq ||\boldsymbol{v}|| + ||\boldsymbol{w}||$.

The most useful multiplication of vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is their **dot product** $\boldsymbol{v} \cdot \boldsymbol{w}$. We multiply the first components $v_1 w_1$ and the second components $v_2 w_2$ and so on. Then we **add those results** to get a single number $\boldsymbol{v} \cdot \boldsymbol{w}$:

$$\boxed{\text{The dot product of } \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \text{ and } \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \text{ is } \boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2.} \qquad (1)$$

If the vectors are in n-dimensional space with n components each, then

$$\textbf{Dot product} \qquad \boxed{\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \boldsymbol{w} \cdot \boldsymbol{v}} \qquad (2)$$

The dot product $\boldsymbol{v} \cdot \boldsymbol{v}$ tells us the squared length $||\boldsymbol{v}||^2 = v_1^2 + \cdots + v_n^2$ of a vector. In two dimensions, this is the Pythagoras formula $a^2 + b^2 = c^2$ for a right triangle. The sides have $a^2 = v_1^2$ and $b^2 = v_2^2$. The hypotenuse has $||\boldsymbol{v}||^2 = v_1^2 + v_2^2 = a^2 + b^2$.

To reach n dimensions, we can add one dimension at a time. Figure 1.2 shows $\boldsymbol{v} = (1, 2)$ in two dimensions and $\boldsymbol{w} = (1, 2, 3)$ in three dimensions. Now the right triangle has sides $(1, 2, 0)$ and $(0, 0, 3)$. Those vectors add to $\boldsymbol{w}$. The first side is in the xy plane, the second side goes up the perpendicular z axis . For this triangle in 3D with hypotenuse $\boldsymbol{w} = (1, 2, 3)$, the law $a^2 + b^2 = c^2$ becomes $(1^2 + 2^2) + (3^2) = 14 = ||\boldsymbol{w}||^2$.

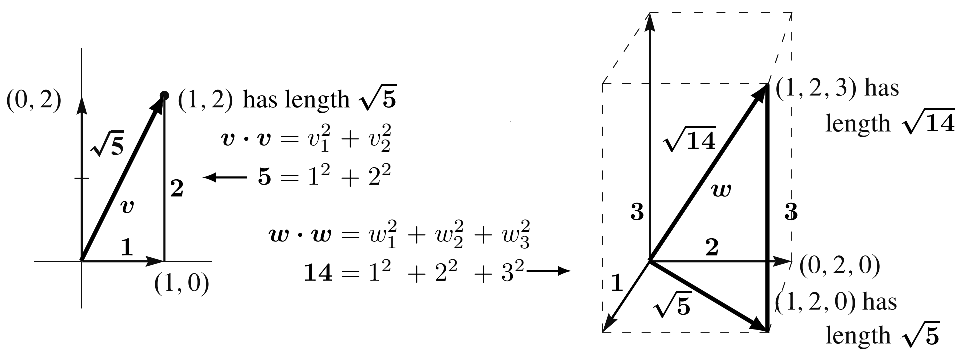


Figure 1.3: The length $\sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{5}$ in a plane and $\sqrt{\boldsymbol{w} \cdot \boldsymbol{w}} = \sqrt{14}$ in three dimensions.

The length of a four-dimensional vector would be $\sqrt{v_1^2+v_2^2+v_3^2+v_4^2}$. Thus the vector $(1,1,1,1)$ has length $\sqrt{1^2+1^2+1^2+1^2} = 2$. This is the diagonal through a unit cube in four-dimensional space. That diagonal in n dimensions has length $\sqrt{n}$.

We use the words **unit vector** when the length of the vector is 1. Divide $\boldsymbol{v}$ by $||\boldsymbol{v}||$.

$$\boxed{\textbf{A unit vector } \boldsymbol{u} \textbf{ has length } ||\boldsymbol{u}|| = \mathbf{1}. \textbf{ If } \boldsymbol{v} \neq \mathbf{0} \textbf{ then } \boldsymbol{u} = \frac{\boldsymbol{v}}{||\boldsymbol{v}||} \textbf{ is a unit vector.}}$$

Example 1 The standard unit vector along the x axis is written $\boldsymbol{i}$. In the xy plane, the unit vector that makes an angle "theta" with the x axis is $\boldsymbol{u} = (\cos\theta, \sin\theta)$:

$$\textbf{Unit vectors} \quad \boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}. \text{ Notice } \boldsymbol{i}\cdot\boldsymbol{u} = \cos\theta.$$

$\boldsymbol{u} = (\cos\boldsymbol{\theta}, \sin\boldsymbol{\theta})$ **is a unit vector because** $\boldsymbol{u}\cdot\boldsymbol{u} = \cos^2\boldsymbol{\theta} + \sin^2\boldsymbol{\theta} = \mathbf{1}$.

In four dimensions, one example of a unit vector is $\boldsymbol{u} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. Or you could start with the vector $\boldsymbol{v} = (1,5,5,7)$. Then $||\boldsymbol{v}||^2 = 1+25+25+49 = 100$. So $\boldsymbol{v}$ has length 10 and $\boldsymbol{u} = \boldsymbol{v}/10$ is a unit vector.

The word "**unit**" is always indicating that some measurement equals "one". The unit price is the price for one item. A unit cube has sides of length one. A unit circle is a circle with radius one. Now we see that a "unit vector" has length $= 1$.

Perpendicular Vectors

Suppose the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ is 90°. Its cosine is zero. That produces a valuable test $\boldsymbol{v}\cdot\boldsymbol{w} = \mathbf{0}$ for perpendicular vectors.

$$\boxed{\textbf{Perpendicular vectors have } \boldsymbol{v}\cdot\boldsymbol{w} = \mathbf{0}. \textbf{ Then } ||\boldsymbol{v}+\boldsymbol{w}||^{\mathbf{2}} = ||\boldsymbol{v}||^{\mathbf{2}} + ||\boldsymbol{w}||^{\mathbf{2}}.} \quad (3)$$

This is the most important special case. It has brought us back to 90° angles and lengths $a^2+b^2=c^2$. The algebra for **perpendicular vectors** $(\boldsymbol{v}\cdot\boldsymbol{w} = 0 = \boldsymbol{w}\cdot\boldsymbol{v})$ is easy :

$$||\boldsymbol{v}+\boldsymbol{w}||^2 = (\boldsymbol{v}+\boldsymbol{w})\cdot(\boldsymbol{v}+\boldsymbol{w}) = \boldsymbol{v}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{v} + \boldsymbol{w}\cdot\boldsymbol{w} = ||\boldsymbol{v}||^2 + ||\boldsymbol{w}||^2. \quad (4)$$

Two terms were zero. Please notice that $||\boldsymbol{v}-\boldsymbol{w}||^2$ is also equal to $||\boldsymbol{v}||^2 + ||\boldsymbol{w}||^2$.

Example 2 The vector $\boldsymbol{v} = (1,1)$ is at a 45° angle with the x axis
The vector $\boldsymbol{w} = (1,-1)$ is at a -45° angle with the x axis
The sum $\boldsymbol{v}+\boldsymbol{w}$ is $(2,0)$. The difference $\boldsymbol{v}-\boldsymbol{w}$ is $(0,2)$.

So the angle between $(1,1)$ and $(1,-1)$ is 90°. Their dot product is $\boldsymbol{v}\cdot\boldsymbol{w} = 1-1 = 0$. This right triangle has $||\boldsymbol{v}||^2 = 2$ and $||\boldsymbol{w}||^2 = 2$ and $||\boldsymbol{v}-\boldsymbol{w}||^2 = ||\boldsymbol{v}+\boldsymbol{w}||^2 = 4$.